THE DUALS OF BERGMAN SPACES IN SIEGEL DOMAINS OF TYPE II

Anatole TEMGOUA KAGOU

Abstract: In every homogeneous Siegel domain of type II, for some real number \( p_0 > 2 \), we characterize the dual of the weighted Bergman space \( A^{p,r} \) when \( 1 \leq p < p_0 \). In the symmetric case, we also characterize the dual of \( A^{p,r} \) with \( p_1 < p < 1 \) for some \( p_1 \in (0,1) \), and extend this to two homogeneous non symmetric Siegel domains of type II.

Résumé : Dans les domaines de Siegel homogènes de type II, nous caractérisons le dual de l'espace de Bergman avec poids \( A^{p,r} \), lorsque \( 1 \leq p < p_0 \), où le nombre réel \( p_0 \) dépend du domaine. Dans le cas où le domaine est symétrique, nous caractérisons également le dual de \( A^{p,r} \) lorsque \( p_1 < p < 1 \), avec \( p_1 \in (0,1) \), et nous généralisons ce résultat à deux domaines de Siegel homogènes et non symétriques de type II.

I. INTRODUCTION.

Let \( D \) be a homogeneous Siegel domain of type II. Let \( dv \) denote the Lebesgue measure on \( D \) and let \( H(D) \) be the space of holomorphic functions in \( D \) endowed with the topology of uniform convergence on compact subsets of \( D \). The Bergman projection \( P \) of \( D \) is the orthogonal projection of \( L^2(D, dv) \) onto its subspace \( A^2(D) \) consisting of holomorphic functions. Moreover, \( P \) is the integral operator defined on \( L^2(D, dv) \) by the Bergman kernel \( B(\zeta, z) \) which, for \( D \), was computed in [G].

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Let r be a real number. Since D is homogeneous, the function \( \zeta \mapsto B(\zeta, \zeta) \) does not vanish on D, we can set:

\[
L^{p,r}(D) = L^p(D, B^{-r}(\zeta, \zeta) \, dv(\zeta)), \quad 0 < p < \infty.
\]

Let p be a positive number. The weighted Bergman space \( A^{p,r}(D) \) is defined by

\[
A^{p,r}(D) = L^{p,r}(D) \cap H(D).
\]

If \( r = 0 \), then \( A^{p,r}(D) \) is simply denoted \( A^p(D) \).

The weighted Bergman projection \( P_e \) is the orthogonal projection of \( L^{2,e}(D) \) onto \( A^{2,e}(D) \). It is proved in [BT] that there exists a real number \( e_D < 0 \) such that \( A^{2,e}(D) = \{0\} \) if \( e \leq e_D \) and that for \( e > e_D \), \( P_e \) is the integral operator defined on \( L^{2,e}(D) \) by the weighted Bergman kernel \( c_e B^{1+e}(\zeta, z) \). In all our work, we shall assume that \( e > e_D \).

The "norm" \( \|f\|_{p,r} \) of \( A^{p,r}(D) \), with \( r > e_D \), is defined by:

\[
\|f\|_{p,r} = \left( \int_D \|B^{-r}(z, z) f(z) \, dv(z) \right)^{1/p}, \quad f \in A^{p,r}(D).
\]

Let \( \rho \) be a positive integer. S.G. Gindikin [G] has defined a differential polynomial \( \Lambda_\rho \) in D that satisfies the property:

\[
\left( \Lambda_\rho \right)_\zeta B(\zeta, z) = c_\rho B^{1+\rho}(\zeta, z) \quad (\zeta, z \in D).
\]

A holomorphic function \( g \) in D is said to be a Bloch function in D if \( g \) satisfies the estimate:

\[
\|g\|_{s} = \sup_{z \in D} \left\{ f(\Lambda_\rho)_z g(z) \left| B^{-\rho}(z, z) \right| \right\} < \infty.
\]

Let \( \mathcal{N} = \{ g \in H(D) : \Lambda_\rho g = 0 \} \). The Bloch space \( \mathcal{B}_\rho \) of D is defined by:
\mathcal{B}_p = \{\text{Bloch functions}\} / \mathcal{N}.

For \( p \geq 1 \), the space \( \mathcal{C}^p_{p,r}(D) \) is the quotient space by \( \mathcal{N} \) of the space of holomorphic functions \( g \) in \( D \) satisfying the estimate

\[
\left( \int_D |B^{-r}_p(z,z)A_p g(z)|^p B^{-r}_p(z,z) \, dv(z) \right)^{1/p} < \infty.
\]

For \( r = 0 \), the space \( \mathcal{C}^p_{p,r}(D) \) is simply denoted \( \mathcal{C}^p_p(D) \).

In the upper half-plane, \( \pi^+ = \{z \in \mathbb{C} : \text{Im} \ z > 0\} \), R. Coifman and R. Rochberg proved the following fact: the dual of the Bergman space \( A^1(\pi^+) \) coincides with the Bloch space of holomorphic functions in \( \pi^+ \), and can be realized as the Bergman projection of \( L^\infty(\pi^+) \). A few years later, D. Békollé in [B4] carried out the same study on symmetric Siegel domains of type II. In fact, he proved that for homogeneous Siegel domains of type II associated with self-dual cones, the dual space of \( A^1 \) coincides with the Bloch space of holomorphic functions, and for symmetric Siegel domains, this space can be realized as the Bergman projection of \( L^\infty \). On the other hand, for bounded symmetric domains, K. Zhu ([Z] and [Z1]) studied the dual of the Bergman spaces \( A^p \) with small exponents \( 0 < p < 1 \) and obtained that their dual is equal to the Bloch space.

Furthermore, in [B6], D. Békollé proved that when \( p \in (1/3 , 4) \), the dual of \( A^p(\mathbb{R}^3 + i\Gamma) \), where \( \Gamma \) is the spherical cone in \( \mathbb{R}^3 \), is equal to \( A^{p'}(\mathbb{R}^3 + i\Gamma) \) ( \( p' \) is the conjugate exponent of \( p \) ) and when \( p \in (1 , 4/3) \), the dual of \( A^p(\mathbb{R}^3 + i\Gamma) \) coincides with the space \( \mathcal{C}^{p'}_p(\mathbb{R}^3 + i\Gamma) \) with \( p = 1 \).

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The purpose of this work is to extend these results to the weighted Bergman space $A^{p,r}(D)$, $1 \leq p < p_0$, in homogeneous Siegel domains of type II, where $p_0$ is a real number greater than 2 and depends on the domain $D$; when $D$ is symmetric, we also show that the dual of $A^{p,r}(D)$, $p_1 < p < 1$, where $p_1$ depends on $D$, is equal the Bloch space. This latter result is also true for two non symmetric domains.

The first aim of this work is to show that there exists $\rho_0 > 0$ such that whenever $\rho > \rho_0$,  

1° when $D$ is homogeneous and $0 < p \leq 1$, $B_\rho$ is isomorphic to a subspace of the dual space $(A^{p,r}(D))^*$ of $A^{p,r}(D)$ and the two spaces are isomorphic when $p = 1$;  

2° the two spaces are equal when $p_1 < p < 1$ when $D$ is symmetric, and the same is true for two particular non symmetric domains.

To show the first claim, let $g$ be in $B_\rho$ and consider the linear functional $\varphi$ on $A^{p,r}(D)$ defined by

$$\varphi(f) = \int_D (\Lambda_\rho) g(z) B^{-\rho}(z,z)f(z)B^{1/p}(z,z)dv(z) \quad (f \in A^{p,r}(D)).$$

Since $\int_D |f(z)|B^{1/p}(z,z)dv(z) < \infty$ for all $f \in A^{p,r}(D)$, it follows that $\varphi$ is bounded, hence belongs to $(A^{p,r}(D))^*$ and is represented by $g$.

Conversely, assume $p = 1$. To obtain $B_\rho = (A^{1,r}(D))^*$, let $\varphi \in (A^{1,r}(D))^*$; then there exists $b \in L^\infty(D)$ such that

$$\varphi(f) = \int_D b(z)f(z)B^{-r}(z,z)dv(z) \quad (f \in A^{1,r}(D)).$$

Since $f(z) = c_{r,\rho} \int_D B^{1/r + \rho}(z,\zeta)f(\zeta)B^{-r}(\zeta,\zeta)dv(\zeta)$, if we set
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\[ g(z) = c_{r,\rho} \int_D B^1 + r + \rho(z,\zeta)B^{-r}(\zeta,\zeta)dv(\zeta), \]

we easily get:

\[ \varphi(f) = \int_D \overline{g(z)f(z)}B^{-r} - \rho(z,\bar{z})dv(z). \]

In fact, since \( \rho \geq \rho_0 \), \( g \) is a holomorphic function on \( D \) satisfying the estimate

\[ \sup_{z \in D} |g(z)|B^{-\rho(z,\bar{z})} \leq c \|h\|_{\infty}. \]

On the other hand, by a lemma of Trèves [Tr], there exists \( \tilde{h} \in \mathcal{H}(D) \) such that \( \Lambda_\rho \tilde{h} = g \).

Let \( h \) be the equivalence class of all holomorphic solutions of this equation. Then \( h \) belongs to \( \mathcal{B}_\rho \), and the equality

\[ \varphi(f) = \int_D \Lambda_\rho h(z)f(z)B^{-r} - \rho(z,\bar{z})dv(z) \]

yields the equality \( (A^{1,r}(D))^* = \mathcal{B}_\rho \).

We next assume that \( 0 < p < 1 \). Let us now define two homogeneous Siegel domains \( D_0 \) and \( D_1 \). Set

\[ V_0 = \left\{ \lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{12} & \lambda_{22} & 0 \\ \lambda_{13} & 0 & \lambda_{33} \end{pmatrix} : \lambda_{11} - \frac{\lambda_{12}^2}{\lambda_{22}} - \frac{\lambda_{13}^2}{\lambda_{33}} > 0, \lambda_{22} > 0, \lambda_{33} > 0 \right\}. \]

Observe that \( V_0 \) is a non self dual cone of rank 3. Define \( D_0 \) by \( D_0 = \mathbb{R}^5 + iV_0 \). Then \( D_0 \) is a homogeneous, non symmetric, tubular domain. On the other hand, \( D_1 \) is the first example, due to Piateckii-Chapiro, of a homogeneous non symmetric Siegel domain of type II, and is defined as follows. Let \( V_1 = \left\{ \lambda = (\lambda_{11}, \lambda_{12}, \lambda_{22}) \in \mathbb{R}^3 : \lambda_{22} > 0, \lambda_{11} - \frac{\lambda_{12}^2}{\lambda_{22}} > 0 \right\} \) be the
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spherical cone in $\mathbb{R}^5$, and consider the $V_1$-Hermitian form $F_1$ in $\mathbb{C}$ defined by

$$F_1: \mathbb{C} \times \mathbb{C} \to \mathbb{C}^3$$

$$(u,v) \mapsto (u\bar{v}, 0, 0).$$

Then $D_1 = \left\{ (z, u) \in \mathbb{C}^3 \times \mathbb{C} : \frac{z - \bar{z}}{2i} - F(u, u) \in V_1 \right\}.$

The restrictions on $D$, $p$ and $\rho$ are as follows: $D$ is either a symmetric Siegel domain of type II, or $D = D_0$, or $D = D_1$. Then there exists $p_1 \in (0, 1)$ such that for all $p \in (p_1, 1)$, we have $(A^{p,r}(D))^* \subset \mathcal{F}_\rho$. Here are the main steps of the proof. Let $\varphi$ be an element of $(A^{p,r}(D))^*$ and set $\alpha = \rho p + r + 1$, with $\rho$ large enough. Let $f \in A^{p,r}(D)$. Then, in view of the molecular decomposition theorem [BT1], there exists $\{\lambda_i\} \in l^p$ such that

$$f(z) = \sum_i \frac{\alpha}{1 + r - \alpha} \lambda_i B^p (z, \bar{z}_i) (z_i, \bar{z}_i),$$

where $\{z_i\}$ is a lattice in $D$. Then we get

$$\varphi(f) = \int_D \varphi(B^p (z, z) f(z)) (z, z) dv(z).$$

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Now, since \( \mathcal{B}(\varphi, z) \) is holomorphic, then by the same lemma of Trèves, there exists a function \( g \in \mathcal{B}_p \) such that \( \Lambda_p g(z) = \varphi \mathcal{B}(\cdot, z) \) by the choice of \( \alpha \). Hence \( \varphi \) is represented by a Bloch function.

To go further, let \( h \in L^\infty(D) \). Take \( \rho \) large enough; then the function \( z \mapsto G(z) = \int_D \mathcal{B}^{1+p}(z, \zeta)h(\zeta)d\nu(\zeta) \) satisfies the estimate \( \sup_{z \in D} \left\{ G(z) \mathcal{B}^{-p}(z, z) \right\} \leq \|h\|_{L^\infty} \) and \( G \) is holomorphic in \( D \). Hence, by the same lemma of Trèves, there exists a function \( g \in \mathcal{B}_p \) such that

\[
(1) \quad (\Lambda_p)g = G.
\]

Now, let \( P \) be the operator from \( L^\infty(D) \) into \( \mathcal{B}_p \) which to each \( h \in L^\infty(D) \) assigns the element \( g = Ph \) defined by (1). This operator \( P \) is called "Bergman projection" of \( L^\infty(D) \) into \( \mathcal{B}_p \) for the following reason: although \( P \) is not the integral operator \( \mathcal{P} \) which is associated with the Bergman kernel \( \mathcal{B}(\zeta, z) \), which has no meaning on \( L^\infty(D) \), it is easy to show that for \( h \in L^2 \cap L^\infty(D) \), the element \( Ph \) of \( \mathcal{B}_p \) can be represented by \( \mathcal{P}h \).

The second aim of this paper is to show that the "Bergman projection" \( P \) is bounded from \( L^\infty(D) \) onto \( \mathcal{B}_p \). In order to achieve this goal, we prove that for each real number \( \rho \) sufficiently large and for each \( G \in H(D) \) such that

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\[ \sup_{z \in D} \left\{ |G(z)|B^{-p}(z, z) \right\} < \infty, \]

one has the reproducing formula:

\[ G(\zeta) = c_p \int_D B^{1+p}(\zeta, z)G(z)B^{-p}(z, z)dv(z) \quad (z \in D). \]

Hence, for each \( g \in \mathcal{B}_p \), if we set \( h(z) = (\Lambda_p)g(z)B^{-p}(z, z) \), then \( h \in L^p(D) \) and \( Ph = g \).

The third aim of this work is to determine on a particular Siegel domain \( D_2 \) of type II, a kernel \( K(\zeta, z) \) that determines \( P \) in the following way: for each \( h \in L^p(D_2) \), a representative of the element \( Ph \) of \( \mathcal{B}_p \) is given by the function \( \zeta \mapsto \int_D K(\zeta, z)h(z)dv(z) \). D. Békollé has determined such a kernel \( K \) in three different Siegel domains of type II, namely, the Cayley transform of the unit ball in \( \mathbb{C}^n \) \([B_1]\), the tube over the spherical cone \([B_2]\) and \([B_3]\), and finally, the tube over the cone of symmetric positive-definite matrices \([B_5]\). The domain we consider is

\[ D_2 = \left\{ (z, u) \in M_2 \times M_{r,2}: \frac{z - z^*}{2i} - u^* u \in V \right\}, \]

where \( M_2 \) (resp. \( M_{r,2} \) is the set of complex matrices of order 2 (resp. with \( r \) lines and 2 rows), and \( V \) the cone of Hermitian positive-definite matrices of order 2. We determine a kernel \( B_0 \) such that

\[ (B - B_0)((\zeta, v), (z, u)) \in L^1(D_2, dv(z, u)) \quad ((\zeta, v) \in D_2); \]

\[ \left( \Lambda_p \right)_\zeta B_0((\zeta, v), (z, u)) = 0. \]

Thus \( Ph \) can be represented by:

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Unfortunately, for Siegel domains of type II associated with cones of rank greater than 2, the determination of a kernel $B_0$ such that (2) and (3) simultaneously hold seems out of reach.

The fourth aim of this work is to prove that there exists $p_0 \in (2, \infty)$ such that whenever $p'_0 < p < p_0$ ($p'_0$ is the conjugate exponent of $p_0$), the dual of $A^{p,r}(D)$ is equal to $A^{p'_0,r}(D)$, and when $1 < p < p_0$, the dual of $A^{p,r}(D)$ is equal to $C^{p'_0}_{p,r}(D)$ with $p > p_0$.

The plan of this work is as follows. In section II, we recall some preliminary results about affine-homogeneous Siegel domains of type II and we give precise statements of our results. In section III, we prove that $\mathcal{B}_p$ is contained in $(A^{p,r}(D))^*$ when $0 < p \leq 1$, and that $\mathcal{B}_p = (A^{1,r}(D))^*$ (Theorem II.7); under some additional assumptions on $D$, $p$ and $\rho$, we prove that $\mathcal{B}_p = (A^{p,r}(D))^*$ (Theorem II.8). In section IV, we show that $P L^\infty(D) = \mathcal{B}_p$ (Theorem II.9). In section V, we determine a defining kernel $B_0$ of a representative of the Bergman projection of a bounded function in the particular domain $D_0$ (Theorem II.10). In section VI, we prove that $(A^{p,r}(D))^* = A^{p',r}(D)$ if $p'_0 < p < p_0$, and $(A^{p,r}(D))^* = C^{p'}_{p,r}(D)$ if $1 < p < p_0$ (Theorem II.11).

For $p = 1$ and $r = 0$, the above results were first presented in [T]. In the sequel, as usual, the same letter $c$ will denote constants that may be different from each other.
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II. STATEMENTS OF RESULTS.

Let $V \subset \mathbb{R}^n$, $n \geq 3$, be an irreducible, open, convex homogeneous cone which contains no straight line. We first recall the canonical decomposition of $V$ as stated in [G].

NOTATIONS. (i) At the $j$th step, $j = 1, 2, \ldots$, the real line will be denoted by $R_{jj}$; at the $k$th step, $k = 1, 2, \ldots$, $R_k$ will stand for the $n_k$-dimensional Euclidean space.

(ii) Let $\Gamma \subset \mathbb{R}^\sigma$ be a convex homogeneous cone which contains no straight line and let $\varphi$ be a homogeneous $\Gamma$-bilinear symmetric form defined on $\mathbb{R}^\tau \times \mathbb{R}^\tau$. The real homogeneous Siegel domain $P = P(\Gamma, \varphi)$ is defined by

$$P = P(\Gamma, \varphi) = \{(y, t) \in \mathbb{R}^\sigma \times \mathbb{R}^\tau : y - \varphi(t, t) \in \Gamma\}.$$ 

We shall denote by $V(P)$ the homogeneous cone defined by:

$$V(P) = \{(y, t, r) \in \mathbb{R}^\sigma \times \mathbb{R}^\tau \times \mathbb{R} : r > 0, (ry, t) \in P\}.$$ 

In order to describe the canonical decomposition of the cone $V$, we consider at the first step, the cone $V^{(1)} = (0, \infty) \subset R_{11}$. At the second step, we associate with $V^{(1)}$ and with a homogeneous $V^{(1)}$-bilinear symmetric form $\varphi^{(2)}$ defined on $R_2$, the real Siegel domain $P^{(2)} = P\left(V^{(1)}, \varphi^{(2)}\right) \subset R_{11} \times R_2$ and then, the convex cone

$$V^{(2)} = V(P^{(2)}) \subset R_{11} \times R_2 \times R_{22}.$$ 

At the $k$th step, we associate with the cone $V^{(k-1)}$ and with a $V^{(k-1)}$-bilinear symmetric form $\varphi^{(k)}$ defined on $R_k$, a real Siegel domain $P^{(k)} = P\left(V^{(k-1)}, \varphi^{(k)}\right) \subset R_{11} \times R_2 \times \cdots \times R_k$. 

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and the cone

\[ V^{(k)} = V(p^{(k)}) = R_{11} \times R_2 \times R_{22} \times \ldots \times R_k \times R_{kk}. \]

It follows from the results of [G] that every homogeneous cone \( V \) which contains no straight line can be decomposed in the form \( V^{(l)} \) (up to a linear isomorphism). The required number of steps to obtain \( V \) in this form is called the rank \( l \) of \( V \), \( V = V^{(l)} \); this yields the following decomposition of the space \( \mathbb{R}^n \) that contains \( V \):

\[ \mathbb{R}^n = R_{11} \times R_2 \times R_{22} \times \ldots \times R_l \times R_{ll}, \quad n = l + \sum_{i=2}^{l} n_i. \]

Furthermore, the projection \( \varphi_{ij}^{(k)} \) of \( \varphi^{(k)} \) onto \( R_{ii} \) \((i < k)\) is a non-negative form. Thus \( \varphi_{ij}^{(k)} \) is positive definite on a subspace \( R_{ik} \) of \( R_k \) with \( \dim R_{ik} = n_k \). We then have:

\[ R_k = \prod_{i=1}^{k-1} R_{ik}, \quad n_k = \sum_{i=1}^{k-1} n_{ik}. \]

We denote by \( G(V) \) the simply transitive group of linear transformations of \( V \) described in [G]. With respect to its canonical decomposition, the cone \( V \) can be described in the following quantitative manner: let \( x \in V \) and let \( x_j, j = 2, \ldots, l \) (resp. \( x_{ii}, i = 1, \ldots, l \)) denote the projection of \( x \) onto \( R_j \) (resp. \( R_{ii} \)); then there exists a unique transformation \( h \in G(V) \) such that \( (h(x))_j = 0, j = 1, \ldots, l \). We set \( \bar{x} = h(x) \). The functions \( \chi_j \) defined for \( j = 1, \ldots, l \), by \( \chi_j(x) = \bar{x}_{jj}, j = 1, \ldots, l \), define the cone \( V \) in the following sense: a point \( x \) of \( \mathbb{R}^n \) belongs to \( V \) if and only if \( \chi_j(x) > 0, j = 1, \ldots, l \).

Since the decomposition (4) of \( \mathbb{R}^n \) yields in a natural way the following decomposition of \( \mathbb{C}^n \):

\[ \mathbb{C}^n = C_{11} \times C_2 \times C_{22} \times \ldots \times C_l \times C_{ll}, \]

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the functions \( \chi_j, j = 1, \ldots, l \), can naturally be extended as rational functions on \( \mathbb{C}^n \).

Let \( \rho = (\rho_1, \ldots, \rho_l) \in \mathbb{C}^l \); we define the rational function \( (z)^\rho \) on \( \mathbb{C}^n \) by:

\[
(z)^\rho = \prod_{j=1}^{l} (\chi_j(z))^{\rho_j}, \quad z \in \mathbb{C}^n.
\]

For \( i = 1, \ldots, l \), set \( m_j = \sum_{k > j} n_{kj} \) and \( d_i = -(1 + \frac{n_{ii} + m_i}{2}) \), and let \( d \) denote the vector of \( \mathbb{R}^l \) whose components are \( d_i \). In the sequel, \( e \) will denote the point of \( V \) whose components are \( e_{ii} = 1, \ e_j = 0, \ i = 1, \ldots, l, \ j = 2, \ldots, l \).

Let us recall the definition of the conjugate cone \( V^* \) of \( V \). Consider the inner product \( \langle \cdot, \cdot \rangle \) defined on \( \mathbb{R}^n \) with respect to the canonical decomposition of \( \mathbb{R}^n \) by:

\[
\langle x, y \rangle = \sum_i x_{ii} y_{ii} + 2 \sum_{i < j} \varphi_{ij} (x_i, y_j).
\]

Then \( V^* \) is defined by:

\[
V^* = \left\{ x \in \mathbb{R}^n : \langle x, y \rangle > 0, \forall y \in V - \{0\} \right\}.
\]

The adjoint group \( G^* (V) \) of \( G(V) \) with respect to \( \langle \cdot, \cdot \rangle \) is the simply transitive group of linear transformations of \( V^* \). The cone \( V^* \) is an irreducible, convex, homogeneous cone which contains no straight line, and it is also of rank \( l \).

We shall denote by \( \chi_j^* \) the defining functions of \( V^* \). We have the following:

\[
n_{ij}^* = n_{ji} (V^*) = n_{j-l+1,l+i+1} \quad (1 \leq i < j \leq l).
\]

For \( \rho \in \mathbb{C}^l \), we define \( \rho^* \) by \( \rho_i^* = \rho_{l-i+1} \), \( i = 1, \ldots, l \), and we also define the function

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The Siegel domain of type II, associated with the homogeneous cone $V$ of $\mathbb{R}^n$ and a $V$-Hermitian, homogeneous form $F : \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^n$, is defined by:

$$D = D(V, F) = \left\{ (z, u) \in \mathbb{C}^n \times \mathbb{C}^m : \frac{z-z}{2i} = F(u, u) \in V \right\}.$$ 

The domain $D$ is then an affine-homogeneous domain. Let $F_{ii}$ denote the projection of $F$ onto $\mathbb{C}_i$, and $\mathbb{C}^{(i)}$ the complex subspace of $\mathbb{C}^m$ on which $F_{ii}$ is positive definite. Set $q_i = \dim \mathbb{C}^{(i)}$; then $m = \sum_{i=1}^{l} q_i$ and $\mathbb{C}^m = \prod_{i=1}^{l} \mathbb{C}^{(i)}$. We shall denote by $q$ the vector of $\mathbb{N}^l$ whose components are $q_i$, $i = 1, \ldots, l$.

We now recall the following two expressions of the Bergman kernel $B((\zeta, v), (z, u))$ of $D = D(V, F)$:

**II.1 PROPOSITION [G]**. The Bergman kernel $B((\zeta, v), (z, u))$ of $D$ is given by

$$B((\zeta, v), (z, u)) = c\left( \frac{\zeta - \bar{z}}{2i} - F(v, u) \right)^{2d - q} = c(\nu^*) \exp\left( -<\lambda, \frac{\zeta - \bar{z}}{2i} - F(v, u)> \right)(\lambda^*)^{-d^* + q^*} d\lambda.$$ 

**NOTATIONS.** For $\alpha = (\alpha_1, \ldots, \alpha_f) \in \mathbb{R}^f$, the notation $1 + \alpha$ stands for the vector $(1 + \alpha_1, \ldots, 1 + \alpha_f)$. Let $\alpha' = (\alpha'_1, \ldots, \alpha'_f) \in \mathbb{R}^f$; we set $\alpha \alpha' = (\alpha_1 \alpha'_1, \ldots, \alpha_f \alpha'_f)$ and by
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\( \alpha > \alpha' \), we mean that \( \alpha_i > \alpha_i' \) for all \( i \in \{1, \ldots, l\} \). For \((\zeta,v)\) and \((z,u)\) in \( D \subset \mathbb{C}^n \times \mathbb{C}^m \), we let \( b \) denote the kernel

\[
b((\zeta,v),(z,u)) = \left( \frac{\zeta - z}{2i} - F(v,u) \right)^{2d-q}.
\]

Notice that \( B = cb \). Moreover, \( b^\alpha \) and \( b^{1+\alpha} \), \( \alpha \in \mathbb{R}^l \), will denote the expressions:

\[
b^\alpha((\zeta,v),(z,u)) = \left( \frac{\zeta - z}{2i} - F(v,u) \right)^{(2d-q)\alpha}
\]

and

\[
b^{1+\alpha}((\zeta,v),(z,u)) = \left( \frac{\zeta - z}{2i} - F(v,u) \right)^{2d-q+(2d-q)\alpha}.
\]

Let \( r \) be a vector of \( \mathbb{R}^l \). For \( p \in (0,\infty) \), we set \( \mathcal{L}^{p,r}(D) = \mathcal{L}^p(D, b^{-r}(z,z)dv(z)) \) and define the weighted Bergman space \( \mathcal{A}^{p,r}(D) \) by \( \mathcal{A}^{p,r}(D) = \mathcal{L}^{p,r}(D) \cap \mathcal{H}(D) \). We equip \( \mathcal{A}^{p,r}(D) \) with the \( \mathcal{L}^{p,r}(D) \) - "norm" \( \| \cdot \|_{p,r} \). The weighted Bergman projection \( P_r \) is the orthogonal projection of \( \mathcal{L}^{2,r}(D) \) onto \( \mathcal{A}^{2,r}(D) \). Recall (cf. [BT]) that \( \mathcal{A}^{2,r}(D) = \{0\} \) when

\[r_i \leq -\frac{n_i + 2}{2(2d-q)i} \quad \text{for some} \quad i \in \{1, \ldots, l\} \quad \text{and otherwise}, \quad P_r \text{ is equal to the integral operator defined on } \mathcal{L}^{2,r}(D) \text{ by the weighted Bergman kernel } c_r b^{1+r}((\zeta,v),(z,u)).
\]

Let us now state some prerequisite results:

**II.2 Theorem** [BT]. Let \( \alpha \) and \( \epsilon \) be in \( \mathbb{R}^l \) and \((\zeta,v)\in D\). Then we have:

\[
\int_{D} b^{1+\alpha}((\zeta,v),(z,u)) b^{-\epsilon}((z,u),(z,u)) dv(z,u) < \infty
\]
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if and only if \( \varepsilon_i > \frac{n_i + 2}{2(2d - q)i} \) and \( \alpha_i - \varepsilon_i > \frac{n_i}{-2(2d - q)i} \), \( i = 1, \ldots, l \). In this case, the following equality holds:

\[
\left[ D \right] b^{1\alpha}(((\zeta, v),(z, u))) b^{-\varepsilon}(z, u)(z, u)dv(z, u) = c_{\alpha, \varepsilon} b^{\alpha - \varepsilon}((\zeta, v),(\zeta, v)).
\]

We shall need the following reproducing formulas which, indeed, improve those obtained in [BT] (Theorem II.6.1, p.225), and whose proof, based on ideas of [BBR], will be given in the appendix:

II.3 THEOREM. Let \( r \) be a vector of \( \mathbb{R}^l \) such that \( r_i > \frac{n_i + 2}{2(2d - q)i} \) for all \( i = 1, \ldots, l \) and \( p \) a real number such that \( 1 \leq p < \min \left\{ \frac{n_i - 2(2d - q)i(1 + r_i)}{n_i} \right\} \). Then for all \( \varepsilon \in \mathbb{R}^l \) such that

\[
\varepsilon_i > \frac{n_i + 2}{2(2d - q)i} \frac{p - 1}{p} + \frac{r_i}{p} \quad (i = 1, \ldots, l),
\]

the reproducing formula \( P_{\varepsilon} f = f \) holds for all \( f \in A^{p,r}(D) \).

II.4 PROPOSITION [BT]. Let \( \alpha \in \mathbb{R}^l \) be such that \( \alpha_i \geq 0 \), \( i = 1, \ldots, l \). Then:

\[
\left| b^{\alpha}(\zeta, v), (z, u) \right| \leq c_{\alpha} b^{\alpha}(\zeta, v), (\zeta, v))
\]

and

\[
\left| b^{\alpha}(\zeta, v) + (\zeta', v'), (z, u) + (z', v') \right| \leq c_{\alpha} b^{\alpha}(\zeta, v), (\zeta, v))
\]

for all \((\zeta, v), (\zeta', v'), (z, u)\) and \((z', v')\) in \( D \).

II.5 LEMMA [R]. For all \( f \in A^{p,r}(D) \) \( (p > 0) \), we have the estimate:
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\[ |f(z,u)|^p \leq c b^{1+r} ((z,u),(z,u)) \|f\|_{p,r}^p. \]

**DEFINITION**: A vector \( \rho \in \mathbb{R}^l \) is a \( V \)-integral vector if \( (\lambda)_p \) is a polynomial in \( \lambda \).

In the sequel, \( \rho \) will be a \( V \)-integral vector. We associate to \( \rho \) the differential polynomial \( (\Lambda_\rho) \zeta \) in \( \mathbb{C}^n \) in the following way: for \( \lambda \in \mathbb{C}^n \),

\[ (\Lambda_\rho) \zeta \exp(\langle \lambda, \zeta \rangle) = (\lambda)_p^* \exp(\langle \lambda, \zeta \rangle) \quad (\zeta \in \mathbb{C}^n). \]

Let us now recall the following lemma due to F. Trèves [Tr] which is crucial in our work.

**II.6 LEMMA** [Tr]. For each holomorphic function \( G \) in \( D \), there exists a holomorphic function \( g \) in \( D \) such that \( (\Lambda_\rho) \zeta g(\zeta, v) = G(\zeta, v) \) for all \( (\zeta, v) \in D \).

**DEFINITION**: A function \( g \in H(D) \) is a Bloch function if:

\[ \|g\|_\sigma = \sup_{(z,u) \in D} \left\{ (\Lambda_\rho) \zeta g(z,u) \left| b^{\sigma - 2d} d ((z,u),(z,u)) \right| < \infty. \right. \]

Set \( \mathcal{N} = \{ g \in H(D) : (\Lambda_\rho)g = 0 \} \). We define the Bloch space \( \mathcal{B}_p \) and the space \( \mathcal{C}_{p,r}(D) \) in the following manner, where we set \( \sigma = \frac{\rho}{-2d + q} \):

\[ \mathcal{B}_p = \{ \text{Bloch functions in } D \}/\mathcal{N}, \]

while \( \mathcal{C}_{p,r}(D) \) is the quotient space:

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\[
\left\{ f \in \mathcal{H}(D) : \| f \|_{C^p(D)} = \left\{ \int_D \left| (\Lambda_\rho)_{z} f(z,u) \right|^p b^{-\rho \sigma - r} ((z,u),(z,u)) \, dv(z,u) \right\}^{1/p} < \infty \right\}
\]

These two spaces have the following topological property:

**LEMMA:** \((\mathfrak{B}, \| \cdot \|)\) and \((C^p_r(D), \| \cdot \|)\) are complex Banach spaces.

Our first two results read as follows:

**II.7 THEOREM.** Let \( D \) be a homogeneous Siegel domain of type II. Let \( p \) be a real number and \( r \) a vector of \( \mathbb{R}^l \) such that \( 0 < p \leq 1 \) and \( r_1 > \frac{n_i + 2}{2(2d - q)i} \), \( i = 1, \ldots, l \). Then the following assertions hold:

(i) \( \mathfrak{B} \) is isomorphic to a subspace of \( (A^{1,r}(D))^\ast \).

(ii) \( \mathfrak{B} \) is equal to \( A^{1,r}(D) \) if \( \rho_1 > \frac{n_1}{2}, i = 1, \ldots, l \) with equivalent norms.

The duality \( (A^{1,r}(D), \mathfrak{B}) \) is given by

\[
(f,g) = \int_D (\Lambda_\rho)_{z} g(z,u) b^{-\sigma} ((z,u),(z,u)) f(z,u) b^{-r} ((z,u),(z,u)) \, dv(z,u),
\]

where \( \sigma = -\frac{\rho}{-2d + q} \).
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**II.8 THEOREM**: Let \( D \subseteq \mathbb{C}^N \) be either a symmetric Siegel domain of type II, or \( D = D_0 \), or \( D = D_1 \). Let \( p_1 = \frac{2N}{2N+1} \) and let \( r \) be a vector of \( \mathbb{R} \) such that \( r_1 > \frac{n_i + 2}{2(2d - q)} \), \( i = 1, \ldots, l \).

Then for all \( p \in (p_1, 1) \) and all \( \rho \in \mathbb{R}^l \) such that
\[
\rho_1 > \frac{n_i + 2}{2} + (-2d + q)_i + \frac{1 + 2n_i}{p} (-2d + q)_i, \quad i = 1, \ldots, l,
\]
we have \( \mathcal{B}_\rho = (A_{p, r}^D(D))^\ast \), with equivalent norms.

Moreover, the duality \( (A_{p, r}^D(D), \mathcal{B}_\rho) \) is given by
\[
(f, g) = \int_D \left( \frac{1 + r}{p} \right)^{\frac{1 + r}{p}} ((z, u), (z, u))^{1/p} d\nu(z, u),
\]
with \( \sigma = \frac{\rho}{-2d + q} \).

**REMARK**: In the symmetric case, R.R. Coifman and R. Rochberg ([CR], p. 43-44) stated that \( A_{p, r}^D(D) \) and \( A_{p_1, 1}^D(D) \) have the same dual. The proof of Theorem II.8 relies on their atomic decomposition theorem. For a proof of the atomic decomposition theorem, cf. [BT1].

Theorems II.7 and II.8 will be proved in section III.

Let \( h \in L^\infty(D) \) and let \( \rho \) be a \( V \)-integral vector such that \( \rho_1 > \frac{n_i}{2} \) \( (i = 1, \ldots, l) \). Then, in view of Theorem II.2, the following function \( G \) is holomorphic in \( D \):
\[
G(\zeta, \nu) = \int_D b^{1+\sigma} ((\zeta, \nu), (z, u)) h(z, u) d\nu(z, u),
\]

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where \( \sigma = \frac{\rho}{-2d + q} \). By Lemma II.6, there exists \( \tilde{g} \in H(D) \) such that \((\Lambda_{\rho})_D \tilde{g}(z, u) = G(z, u)\). Let \( g \) be the equivalence class of all holomorphic solutions of this equation. Then \( g \in \mathfrak{B}_{D} \); hence, we can define an operator \( P \) from \( L^\infty(D) \) into \( \mathfrak{B}_{D} \) in the following way:

\[
Ph = g.
\]

\( P \) is called the "Bergman projection" of \( L^\infty(D) \) into \( \mathfrak{B}_{D} \). Let us justify this name: the Bergman projection \( P \) is usually the integral operator defined on \( L^2(D) \) by:

\[
P_h(\zeta, \nu) = \int_D B((\zeta, \nu), (z, u)) h(z, u) dv(z, u) \quad (h \in L^2(D)).
\]

Now, let \( h \in L^2 \cap L^\infty(D) \); since

\[
(\Lambda_{\rho})_D b((\zeta, \nu), (z, u)) = c_\rho b^{1 + \sigma}((\zeta, \nu), (z, u))
\]

(recall \( \sigma = \frac{\rho}{-2d + q} \)), one easily obtains that \( P h \) is a representative of the element \( Ph = g \) of \( \mathfrak{B}_{D} \).

Our third result reads as follows:

**II.9 THEOREM.** Let \( D \) be a homogeneous Siegel domain of type II. Let \( \rho \) be a \( V \)-integral vector such that \( \rho_i > \frac{n_1}{2} \) \((i=1, \ldots, l)\). Then \( PL^\infty(D) = \mathfrak{B}_{D} \) and \( P \) has a bounded right inverse.

Theorem II.9 will be proved in section IV.

Our fourth result reads as follows:
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II-10 THEOREM: Let \( D_2 = \left\{ (z,u) \in M_2 \times M_r \mid \frac{z \cdot z^*}{2i} - u^* u \in V_2 \right\} \) and \( \rho = (5,5) \). Then there exists a kernel \( b_0 \) in \( D_2 \) such that:

(i) with respect to \( (\zeta, v) \), \( b_0 ((\zeta, v), (z,u)) \) is holomorphic in \( D_2 \) and \( (\Lambda_\rho) \zeta b_0 ((\zeta, v), (z,u)) = 0 \),

(ii) for all \( (\zeta, v) \in D_0 \), \( (b- b_0) ((\zeta, v), (z,u)) \in L_1 (D_2, dv(z,u)) \).

Hence, for each \( h \in L^{\infty} (D_2) \), the function \( \mathcal{P} h \) defined on \( D_2 \) by

\[
\mathcal{P} h(\zeta, v) = \int_{D_2} (b - b_0)((\zeta, v), (z,u)) h(z,u) dv(z,u)
\]

is a representative of \( \mathcal{P} h \).

This result will be proved in section V.

Our fifth result focuses on the case \( p > 1 \) and reads as follows:

II-11 THEOREM: Let \( D \) be a homogeneous Siegel domain of type II. Let \( p \) be a real number and \( r \) a vector of \( \mathbb{R}^l \) such that \( \gamma_i = \frac{n_i + 2 - 2(d - q) j_i}{2(d - q) n_i} \) \((i = 1, \ldots, l) \) and \( 1 < p < \min_i \left\{ \frac{2n_i + 2 - 2(d - q) j_i}{n_i} \right\} \).

Then we have the following assertions.

(i) If \( \max_i \left\{ \frac{2n_i + 2 - 2(d - q) j_i}{n_i + 2 - 2(d - q) j_i} \right\} < p < \min_i \left\{ \frac{2n_i + 2 - 2(d - q) j_i}{n_i} \right\} \), then \( (A^{p,r})^* = A^{p',r} \) \((D)\) \((p' \) is the conjugate exponent of \( p \)).
(ii) \((A^{p,r}(D))^* = \mathfrak{C}_{p,r}^p(D)\) whenever \(p_i > \frac{n_i}{2}\), \(i = 1, \ldots, l\), with equivalent norms.

The duality \((A^{p,r}(D), \mathfrak{C}_{p,r}^p(D))\) is given by

\[
(f, h) = \int_D b^{-\sigma}((z,u),(z,u))^{\left(\frac{1}{p} + \frac{1}{q} - 1\right)} b^{-r} f((z,u), (z,u)) \, d\nu(z,u),
\]

where \(\sigma = \frac{p}{-2d + q}\).

This theorem will be proved in section VI.

**III. PROOFS OF THEOREMS II.7 AND II.8.**

**III.1 PROOF OF THEOREM II.7**

(i) Let \(g \in \mathfrak{S}_p\) and consider the linear functional \(\varphi\) defined on \(A^{p,r}(D)\) by:

\[
\varphi(f) = \int_D \left(\frac{1}{p} - 1\right) b^{-r} (\zeta, \nu) \, d\nu (\zeta, \nu),
\]

where \(\sigma = \frac{p}{-2d + q}\) and \(0 < p \leq 1\). By Lemma II.5, we have the estimate:

\[
\int_D \left(\frac{1}{p} - 1\right) b^{-r} (\zeta, \nu) \, d\nu (\zeta, \nu) \leq c \left( \int_D \|f(\zeta, \nu)\|_{p,r}^p b^{-r} \, d\nu (\zeta, \nu) \right)^{\left(\frac{1}{p} - 1\right)} \|f\|_{p,r}.
\]

Hence \(\varphi\) possesses the following property:

\[
|\varphi(f)| \leq c \|g\| \|f\|_{p,r}.
\]

and thus \(\|\varphi\| \leq c \|g\|\). Therefore \(\varphi \in (A^{p,r}(D))^*\). This proves the inclusion of \(\mathfrak{S}_p\) into

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\((A^{1,r}(D))^*\).

(ii) Let \( p = 1 \), and let \( \varphi \) be in \((A^{1,r}(D))^*\). Then by the Hahn-Banach theorem, there exists a bounded function \( k \) in \( D \) such that

\[
\varphi(f) = \int_D k(z,u)f(z,u)b^{-r}(z,u)dv(z,u).
\]

On the other hand, by Theorem II.3, we have the following reproducing formula for every \( f \in A^{1,r}(D) \):

\[
f(z,u) = \int_D b^{1+r} f((z,u),(\zeta,v))b^{-r}(\zeta,v)dv(\zeta,v).
\]

Therefore by the Fubini theorem, we easily get

\[
\varphi(f) = \int_D \left( \int_D b^{1+r} f((z,u),(\zeta,v))b^{-r}(\zeta,v)dv(\zeta,v) \right) dv(z,u).
\]

Set \( g(z,u) = \int_D b^{1+r} f((z,u),(\zeta,v))b^{-r}(\zeta,v)dv(\zeta,v) \). Observe next that \( g \in H(D) \) and that by Theorem II.2, the following estimate holds:

\[
|g(z,u)| \leq C \|k\|_\infty b^\sigma((z,u),(z,u)) \quad ((z,u) \in D),
\]

since \( \sigma_i = \frac{n_i}{2(2d-q)i} \). Then in view of Lemma II.6, there exists \( h \) in \( \mathcal{B}_p \) such that \( \Lambda_p h = g \). Therefore

\[
\varphi(f) = \int_D \left( \Lambda_p f(z,u)b^{-r}(z,u)dv(z,u) \right)
\]

and thus, \( \varphi \) is represented by the element \( g \) of \( \mathcal{B}_p \). This proves the reverse inclusion of \( (A^{1,r}(D))^* \) in \( \mathcal{B}_p \).
We shall now prove another reproducing formula whose proof relies on the dominated convergence theorem. In view of this formula, the Bergman weighted projection $P_r$ also reproduces functions which satisfy certain uniform estimate.

III.2 PROPOSITION: Let $r$ and $\varepsilon$ be two vectors of $\mathbb{R}^l$ such that

$$v_i > \frac{n_i}{2(2d-q_i)} \quad \text{and} \quad \tau_i > \frac{n_i+2}{2(2d-q_i)} + v_i, \quad i = 1, \ldots, l.$$ Let $G$ be in $H(D)$ such that

$$\sup_{z \in D} \left\{ |G(z)| b^{-\varepsilon}(z,z) \right\} < \infty.$$ Then $P_r G = G$.

PROOF: Consider the sequence $\{G_n\}$ defined by

$$G_n(z) = G\left(z + \frac{\varepsilon}{n}\right) b^{\alpha}\left(\frac{z + \varepsilon}{n}\right),$$

where the positive exponent $\alpha$ is to be specified later. There exists $c > 0$ such that for every $z$ in $D$ and every positive integer $n$:

$$\left| G\left(z + \frac{\varepsilon}{n}\right) \right| \leq cb^{\varepsilon}\left(\frac{z + \varepsilon}{n}\right),$$

where the latter inequality is yielded by Proposition II.4 since $v_i > 0$. Then, when we keep $n$ fixed, the function $z \mapsto G\left(z + \frac{\varepsilon}{n}\right)$ is bounded and the following inequality holds:

$$\int_D \left| G_n(z)\right|^2 b^{-r}(z,z) dv(z) \leq C_n \int_D \left| b^{2\alpha}\left(\frac{z + \varepsilon}{n}\right)\right|^2 b^{-r}(z,z) dv(z).$$
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We choose \( \alpha \) so that \( 2\alpha_i - 1 - r_i > \frac{n_i}{-2(2d - q)} \) for all \( i = 1, \ldots, l \). Hence by Theorem II.2, the latest integral converges and furthermore, \( P_r G_n = G_n \) for all \( n \).

On the other hand, by Proposition II.4, since \( \alpha_i > 0 \) and \( r_i > 0 \), we have:

\[
\left| G_n(z) \right| b^{-\frac{r}{2}}(z, z) \leq \left| G(z + \frac{ic}{n}) \right| b^{-\frac{r}{2}}(z, z) \leq c.
\]

Also, by Theorem II.2, under our assumptions \( r_i > \frac{n_i + 2}{2(2d - q)} \) and

\[
\eta_i \geq \frac{n_i + 2}{2(2d - q)} + r_i, \quad i = 1, \ldots, l,
\]

we get:

\[
\int_D \left| b^{1 + r} (z, \zeta) \right| b^{-r + \frac{r}{2}} (\zeta, \zeta) dv(\zeta) < \infty.
\]

Henceforth, by the dominated convergence theorem, we easily obtain the equality \( P_r G = G \).

**REMARK** : This result extends the one obtained in [B2] for symmetric Siegel domains of type II, via the bounded circular realization of the domain. Our proof is straight and includes all homogeneous Siegel domains of type II.

**III.3 COROLLARY**: Let \( p \in (0, 1) \). Let \( \alpha \) and \( r \) be vectors of \( \mathbb{R}^l \) such that

\[
\alpha_i > \frac{n_i}{-2(2d - q)} + 1 + r_i \quad \text{and} \quad r_i > \frac{n_i + 2}{2(2d - q)} \quad \text{for all} \quad i = 1, \ldots, l.
\]

Then for all \( \varphi \in (A^{P_r(D)})^* \) and for all \( z_0 \) in \( D \), we have

\[
\int_D b^P (z_0, z) \varphi \left( \frac{\alpha}{b^P (z, z)} \right) b^{1 - \frac{\alpha}{2}} (z, z) dv(z) = \varphi \left( \frac{\alpha}{b^P \left( \cdot, z_0 \right)} \right).
\]
PROOF: By Proposition III.2, it is sufficient to show that
\[ \sup_{z \in D} \left| \frac{\alpha}{b^p(z, z)} \right| \frac{1 + r - \alpha}{b^p} (z, z) < \infty. \]

But since \( \varphi \in (A^{p,r}(D))^* \), we obtain that
\[ \left| \frac{\alpha}{b^p(z, z)} \right| \leq c \left| \frac{\alpha}{b^p(z, z)} \right| \leq c \cdot b^{1 - r} (z, z), \]

where the latest estimate follows from Theorem II.2. This proves Corollary III.3.

\[ \square \]

III.4 PROOF OF THEOREM II.8

Let \( D \subset \mathbb{C}^N \) be a symmetric Siegel domain of type II, or let \( D \) be equal to \( D_0 \) or \( D_1 \). Remark that for \( p \in \left( \frac{2N}{2N+1}, 1 \right) \) and \( r \in \mathbb{R}^l \) satisfying \( r_1 > \frac{n_i + 2}{2(2d - q_i)} \), the domain \( D \) satisfies the hypotheses of the molecular decomposition theorem [BT1] for functions in Bergman spaces \( A^{p,r}(D) \); more precisely, for such values of \( p \) and \( r \), there exist constants \( c = c(p, r) \) and \( C = C(p, r) \) such that for every \( f \in A^{p,r}(D) \), there exists an \( l^p \)-sequence \( \{\lambda_i\} \) such that
\[ f(z) = \sum_{i=0}^{\infty} \lambda_i b^p(z, z_i) \frac{1 + r - \alpha}{b^p} (z_i, z_i) \quad (z \in D), \]

where \( \{z_i\} \) is a lattice in \( D \) and the following estimate holds:
\[ \|f\|_{l^p} \leq \|\lambda_i\| < C\|f\|_{p,r} \].

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Moreover, the vector $\alpha$ of $R^l$ is defined as follows. Let $\rho$ denote a $V$-integral vector in $R^l$ such that when we set $\sigma = \frac{\rho}{-2d + q}$, the following condition is satisfied:

$$\sigma_i > \frac{n_i + 2}{2(2d - q)i} + 1 + \frac{1 + 2n_i}{p} \quad i = 1, ..., l.$$  

The vector $\alpha$ is given by $\alpha = \sigma p + 1 + r$.

Let $\varphi \in (A^{p,r}(D))^*$. For $f \in A^{p,r}(D)$, define the sequence $\{f_N\}$ of functions in $D$ by

$$f_N(z) = \sum_{i=0}^{N} \frac{\alpha}{\lambda_i} \left( \frac{1 + r - \alpha}{p} \right)^{N+1} b^{P}(z, z_i) b^{p}(\cdot, z_i) \quad (z \in D).$$  

Then $\{f_N\}$ converges to $f$ in $A^{p,r}(D)$. Hence $\varphi(f_N)$ goes to $\varphi(f)$ as $N$ tends to infinity.

Now, in view of Corollary III.3, we get:

$$\varphi(f_N) = \int_D f_N(z) \varphi \left( b^{P}(\cdot, z) \right) |dz|^{p-1} = \int_D f_N(z) \varphi \left( b^{P}(\cdot, z) \right) |dz|^{p-1}$$

and combining this with (7) yields for every positive integer $N$:

$$\varphi(f_N) = \int_D f_N(z) \varphi \left( b^{P}(\cdot, z) \right) |dz|^{p-1}$$

This easily implies that for every positive integer $N$:

$$(8) \quad \varphi(f_N) = \int_D f_N(z) \varphi \left( b^{P}(\cdot, z) \right) |dz|^{p-1}$$

Now, let $N$ tend to infinity in identity (8). On the one hand, $\varphi(f_N)$ goes to $\varphi(f)$. On the
other hand, it follows from (6) that
\[
\left| \int_D (f_N - f)(z) \phi \left( \frac{\alpha}{b^P(z)} \right) \left( 1 - \frac{\alpha}{b^P(z)} \right) (z, z) d\nu(z) \right| \leq C \int_D \left| (f_N - f)(z) \right| \left( 1 - \frac{1 + r}{p} (z, z) d\nu(z) \right)
\]
\[
\leq C \int_D \left| (f_N - f)(z) \right| ^p \left( b^{-r}(z, z) d\nu(z) \right)
\]
where the latest inequality follows from Lemma II.5. Henceforth, since \( \{f_N\} \) converges to \( f \) in \( A^{p, r}(D) \), we conclude that the right hand side of (8) tends to
\[
\int_D f(z) \phi \left( \frac{\alpha}{b^P(z)} \right) \left( 1 - \frac{\alpha}{b^P(z)} \right) (z, z) d\nu(z) .
\]
We have then proved that for every \( f \) in \( A^{p, r}(D) \):
\[
\varphi(f) = \int_D f(z) \phi \left( \frac{\alpha}{b^P(z)} \right) \left( 1 - \frac{\alpha}{b^P(z)} \right) (z, z) d\nu(z).
\]

Now, by Lemma II.6, since the function \( z \mapsto \phi \left( \frac{\alpha}{b^P(z)} \right) \) is holomorphic in \( D \), there exists
\[
g \in H(D) \text{ such that } A_{\alpha} g(z) = \phi \left( \frac{\alpha}{b^P(z)} \right).
\]
Furthermore, with the choice of \( \alpha \) and with
\[
\sigma = \frac{\alpha - 1 - r}{p},
\]
we deduce from (6) that for every \( z \in D \), the following estimate holds:
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\[ \sup_{z \in D} \left( \frac{\alpha}{b^p(z)} \right) b^{-\sigma}(z, z) < \infty. \]

Hence \( g \) is a Bloch function in \( D \), and \( g \) clearly represents the bounded linear functional \( \varphi \in (A^{p,r}(D))^* \) in the following manner:

\[ \varphi(f) = \int_D \lambda_p g(z) b^{-\sigma}(z, f(z)) z \, dv(z). \]

Moreover, \( \|\varphi\|_* \leq c\|g\| \). This proves the inclusion of \( (A^{p,r}(D))^* \) in \( \mathcal{B} \). Theorem II.8 is entirely proved.

**IV. PROOF OF THEOREM II.9.**

Since \( \mathcal{PL}^{\infty}(D) \subset \mathcal{B} \), let us prove that \( \mathcal{B} \subset \mathcal{PL}^{\infty}(D) \). Let \( g \in \mathcal{B} \) and set

\[ h(z, v) = \left( \lambda_p \right)_z g(z, v) b^{-\sigma}(z, (z, v)) \quad \text{with} \quad \sigma = \frac{p}{-2d + q}. \]

Then \( h \in L^{\infty}(D) \). We are going to show that \( Ph = g \). It suffices to prove the equality:

\[ \left( \lambda_p \right)_z g(z, v) = \int_D b^{1+\sigma}((z, v), (z, u)) \left( \lambda_p \right)_z g(z, u) b^{-\sigma}(z, (z, u)) \, dv(z, u). \]

Setting \( \varepsilon = \sigma \), this equality follows from Proposition III.2. To complete the proof, let us determine a right inverse of \( P \). Define \( R \) as follows:

\[ Rg(z, v) = \left( \lambda_p \right)_z g(z, v) b^{-\sigma}(z, (z, v)) \quad \text{for} \quad g \in \mathcal{B}. \]

Then \( PRg = g \) and \( Rg \in L^{\infty}(D) \). Furthermore, \( \|Rg\|_{L^{\infty}(D)} \leq \|g\|_* \) and thus \( \|R\| \leq 1 \). This completes the proof of Theorem II.9. \( \Box \)
V. PROOF OF THEOREM II.10.

V.1 Preliminaries on $D_2$.

The cone $V = \{ z \in M_2, z \text{ Hermitian, } z > 0 \}$ is of rank 2 and linearly equivalent to the spherical cone in $\mathbb{R}^4$; moreover, it is self-conjugate with respect to the inner product

$$<\zeta, z> = \zeta_{11} z_{11} + \zeta_{12} z_{12} + \zeta_{21} z_{21} + \zeta_{22} z_{22},$$

where $\zeta = \begin{pmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & \zeta_{22} \end{pmatrix}$ and $z = \begin{pmatrix} z_{11} \\ z_{21} \\ z_{12} \\ z_{22} \end{pmatrix}$ belong to $M_2$. Set $e = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. We also have $n_{12} = 2$, $n_1 = 0$, $n_2 = 2$, $d = (-2, -2)$. The functions that define the cone $V$ are defined by

$$\chi_1(\zeta) = \zeta_{11} - \zeta_{12} \sqrt{\zeta_{21}} \zeta_{22}, \quad \chi_2(\zeta) = \zeta_{22} \quad (\zeta \in M_2).$$

Let $F: M_{r,2} \times M_{r,2} \to M_2$ be defined by $F(v,u) = u^* v$ where $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are in $M_{r,2}$ with $v_1, v_2$ in $\mathbb{C}^r$ $(i = 1, 2)$. Hence $F_{11}(v,u) = u_1^* v_1$, $F_{22}(v,u) = u_2^* v_2$ and it is clear that $F_{ii}$ is concentrated on $\mathbb{C}^r \times \mathbb{C}^r$ and is positive definite on $\mathbb{C}^r$ $(i = 1, 2)$. Therefore $q = (r,r)$.

Let $\rho = (5,5)$ be a $V$-integral vector. It is straightforward that

$$\left( \Lambda \rho \right)_\zeta = \left( \begin{array}{cc} \rho^2 & \rho^2 \\ -\rho^2 & -\rho^2 \end{array} \right)^5 \left( \begin{array}{cc} \zeta_{11} \delta \zeta_{22} & \zeta_{12} \delta \zeta_{21} \\ \zeta_{21} \delta \zeta_{11} & \zeta_{22} \delta \zeta_{12} \end{array} \right)$$

and

$$\left( \Lambda \rho \right)_\zeta B((\zeta,v),(z,u)) = c_{\rho} B^{1 + \frac{\rho}{-2d + q}} ((\zeta,v),(z,u)), \quad 69.$$
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where B is the Bergman kernel of D_2. It then follows from Theorem II.2 that
\( (\Lambda_\rho)_{\zeta} B((\zeta,v),(z,u)) \) is integrable with respect to \((z,u)\) in \(D_2\) since \(\rho = (\rho_1, \rho_2)\), \(\rho_1 > 0\), \(\rho_2 > 1\). We are now ready to state the following lemma whose proof is just computational.

**V.2 Lemma**: Let \(\alpha_1\) and \(\alpha_2\) be two integers such that \(\alpha_1 \in \{-4,-3,-2,-1,0\}\) or \(\alpha_2 \in \{-3,-2,-1,0,1\}\). Then \( (\Lambda_\rho)_{\zeta} \chi_2^{-\alpha_2} \chi_1^{-\alpha_1} \left( \frac{\zeta - z^*_{\rho} - u^* v}{2i} \right) = 0 \), where \((z,u)\) and \((\zeta,v)\) belong to \(D_2\).

The proof of the next lemma is somewhat lengthy and will be given in the appendix.

**V.3. Lemma**: The following functions are integrable in \(D_2\):

(a) \[ |u_2|^4 \chi_1^{-5-r} \chi_2^{-8+r} \left( \frac{ie - z^*}{2i} \right) ; \]

(b) \[ |u_2|^5 \chi_1^{-5-r} \chi_2^{-9+r} \left( \frac{ie - z^*}{2i} \right) ; \]

(c) \[ |u_2|^4 z_{12} \chi_1^{-5+r} \chi_2^{-9+r} \left( \frac{ie - z^*}{2i} \right) ; \]

(d) \[ |u_2|^4 z_{21} \chi_1^{-5+r} \chi_2^{-9+r} \left( \frac{ie - z^*}{2i} \right) ; \]

(e) \[ |u_2|^5 z_{21} \chi_1^{-5+r} \chi_2^{-9+r} \left( \frac{ie - z^*}{2i} \right) ; \]
We shall also use the following lemma.

V.4. LEMMA : Let $(\zeta, v)$ be in $D_0$. Then there exists two positive constants $c_1$ and $c_2$ such that for all $(z, u) \in D_2$, the following estimates hold:

\begin{align*}
a) \quad & c_1 \left| \chi_1 \left( \frac{ie - z^*}{2i} \right) \right| \leq \left| \chi_2 \left( \frac{\zeta - z^*}{2i} - u^* v \right) \right| \leq c_2 \left| \chi_2 \left( \frac{ie - z^*}{2i} \right) \right|; \\
b) \quad & c_1 \left| \chi_1 \left( \frac{ie - z^*}{2i} \right) \right| \leq \left| \chi_1 \left( \frac{\zeta - z^*}{2i} - u^* v \right) \right| \leq c_2 \left| \chi_1 \left( \frac{ie - z^*}{2i} \right) \right|.
\end{align*}

PROOF : This is a straightforward consequence of the following lemma due to A. Koranyi:

LEMMA ([CR], cf. also [BT1]) Let $D$ be a symmetric Siegel domain of type II and let $d$ denote the Bergman distance on $D_2$. Then there exists a constant $C_D$ such that for all $\zeta, z, z'$,

\[ \frac{B(\zeta, z)}{B(\zeta, z')} - 1 \leq C_D d(z, z') \] whenever $d(z, z') \leq 10$.

We are now ready to handle the proof of Theorem II.10:
PROOF OF THEOREM II.10. We can define \( b_0 \) in the following manner:

\[
(b - b_0)((z, v), (z, u)) = \begin{pmatrix}
X_1 & -X_1(z + u \ast \nu_v) & (z + u \ast \nu_v)(1) \\
X_2 & -X_2(z + u \ast \nu_v) & (z + u \ast \nu_v)(1)
\end{pmatrix}
\]

\[
X_1 = \sum_{n = 0}^{\infty} \left( \frac{z + u \ast \nu_v}{2i} \right)^n
\]

where \( \alpha = \frac{(4 + r)(5 + r)}{2}, \beta = \frac{(4 + r)(5 + r)(6 + r)}{2} \) and \( \left( \zeta(1), v(1) \right) = \left( \begin{pmatrix} i \\ 0 \end{pmatrix}, (0, v_2) \right) \).

Then \( b_0 ((z, v), (z, u)) \) is a holomorphic function of \((z, v)\) in \( D_2 \). Furthermore, in view of Lemma V.2 and the fact that \( X_1 \left( \frac{z(1) - z \ast \nu_v}{2i} \right) \) does not depend on \( z_{11}, z_{21} \) and \( z_{12} \), we have \( \left( A_{D} \right)_s b_0 ((z, v), (z, u)) = 0 \). This proves the first assertion.

Using the identity \( a^{-(n + 1)} - b^{-(n + 1)} = (b - a) \sum_{k = 0}^{n} a^{-(k + 1)} b^{-(n + 1 - k)} \) and in view of Lemma V.4, we have the estimate:

\[
(b - b_0)((z, v), (z, u)) \leq \left\| X_1^{(5 + r)} X_2^{(8 + r)} \left( \frac{z + u \ast \nu_v}{2i} \right) \right\| \left\| \begin{pmatrix} z_{11} - i \\ z_{12} \end{pmatrix} \right\| \left\| \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right\| \left\| \begin{pmatrix} z_{21} \end{pmatrix} \right\| \left\| \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \right\| + \frac{1}{4} X_2 \left( \frac{z + u \ast \nu_v}{2i} \right)
\]

\[
\left\| ((z_{11} + z_{12})^2 + (z_{21} + z_{22})^2 + (z_{12} + 2z_{11})^2 + (z_{21} + 2z_{22})^2 + 2z_{12}^2 + 2z_{21}^2 + 4u_1^2 + 4v_1^2 \right\|
\]

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Hence in view of Lemma V.3, \((b-b_0) ((z, u), (z, u))\) is in \(L^1(D_2, dv(z, u))\). This completes the proof of Theorem II.10. \(\square\)

VI. PROOF OF THEOREM II.11.

VI.1. THEOREM [BT]. Let \(\varepsilon\) and \(r\) be in \(\mathbb{R}^l\) such that \(\varepsilon_i > \frac{1}{(2d - q)_i}\) and

\[
\eta_i = \frac{n_i + 2 - 2(2d - q)_i}{n_i} , i=1,\ldots,l.
\]

Then \(P_{\varepsilon}\) is bounded from \(L^{p,r}(D)\) into \(A^{p,r}(D)\) if

\[
\max_{i=1,\ldots,l} \left\{ 1, \frac{2n_i + 2 - 2(2d - q)_i \varepsilon_i}{n_i} \right\} < p < \min_{i=1,\ldots,l} \frac{2n_i + 2 - 2(2d - q)_i \varepsilon_i}{n_i}.
\]

I.2 PROOF OF THEOREM II.11. (i) In view of Theorems II.3 and VI.1, we have \(P_r f = f\) for every \(f \in A^{p,r}(D)\), and \(\|P_r g\|_{p,r} \leq c_{p,r}\|g\|_{p,r}\). One easily obtains the announced results from the Hahn-Banach and Riesz representation theorems.

(ii) Let \(\varphi \in (A^{p,r}(D))^*\). By the Hahn- Banach theorem , there exists \(g \in L^{p,r}(D)\) such that for all \(f \in A^{p,r}(D)\), we have

\[
\varphi(f) = \int_D g(z, u) f(z, u) b^{-r}((z, u), (z, u)) dv(z, u).
\]

Set \(\sigma = \frac{\rho}{-2d + q}\). By Theorem II.3, \(P_{\sigma + r} f = f\), \(f \in A^{p,r}(D)\), then

\[
\varphi(f) = \int_D g(z, u) P_{\sigma + r} f(z, u) b^{-r}((z, u), (z, u)) dv(z, u)
\]

\[
= \int_D T_{\sigma + r, r} g(z, u) f(z, u) b^{-r}((z, u), (z, u)) dv(z, u)
\]

where \(T_{\sigma + r, r}\) is the adjoint operator of \(P_{\sigma + r}\) with respect to the inner product

\[
< f, g >_{2, r} = \int_D f(z, u) g(z, u) b^{-r}((z, u), (z, u)) dv(z, u),
\]

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and it is expressed in the following way:

\[ T_{\sigma + r, p}(w) = b^{-\sigma}(w, w) \int_D b^{1 + \sigma + r}(w, (z, u)) f(z, u) b^{-r}((z, u), (z, u)) dv(z, u) \]  

\( (w \in D) \).

Since \( \rho_1 > \frac{n_i}{2} \) and \( p' > \max \left( \frac{2n + 2 - 2(2d - q)_{1}r_1}{n_1 + 2 - 2(2d - q)_{1}r_1}, \frac{2n + 2 - 2(2d - q)_{1}r_1}{n_1 + 2 - 2(2d - q)_{1}r_1} \right) \), it follows from Theorem VI.1 that

\[ \| h_{r, r'} \|_{p', r} \leq c_{p, r} \| \varphi \|_{p', r}. \]

Clearly, the function \( z \mapsto b^{\sigma}(z, z) T_{\sigma + r, r}(z) \) belongs to \( H(D) \), then by Lemma II.6, there exists \( h \in H(D) \) such that \( \Lambda_{\rho} h(z) = b^{\sigma}(z, z) T_{\sigma + r, r}(z) \) for every \( z \in D \). Then \( h \in C_{p', r}(D) \) and \( \| h \|_{C_{p', r}(D)} \leq c \| \varphi \| \). Thus, finally, \( h \) represents \( \varphi \) in the following way:

\[ \varphi(f) = \int_D b^{-\sigma}((z, u), (z, u)) \Lambda_{\rho} h(z, u) f(z, u) b^{-r}((z, u), (z, u)) dv(z, u). \]

Conversely, let \( h \in C_{p', r}(D) \); then the function \( z \mapsto b^{-\sigma}(z, z) \Lambda_{\rho} h(z) \) belongs to \( L^{p', r}(D) \) and the linear functional \( \varphi \) defined by

\[ \varphi(f) = \int_D b^{-\sigma}((z, u), (z, u)) \Lambda_{\rho} h(z, u) f(z, u) b^{-r}((z, u), (z, u)) dv(z, u) \]  

\( (f \in A^{p, r}(D)) \)

is such that \( \| \varphi \| \leq c \| h \|_{C_{p', r}(D)} \). Hence \( \varphi \in (A^{p, r}(D))^* \). Therefore this proves assertion (ii) and Theorem II.11 is entirely proved.

REMARK: Let \( r \) and \( p \) satisfy the hypotheses of assertion (i) of Theorem II.11, and let \( \rho \) be a vector of \( \mathbb{R}^l \) such that \( \rho_1 > \frac{n_i}{2} \) for all \( i = 1, ..., l \). Then \( C_{p', r}(D) \) is equivalent to...
\( A^{p,r}(D) \) in the following manner: every equivalence class in \( \mathcal{C}_{p,r}(D) \) contains one and only one representative \( f \) belonging to \( A^{p,r}(D) \), and conversely, every \( f \in A^{p,r}(D) \) is a representative of an equivalence class in \( \mathcal{C}_{p,r}(D) \). In fact, first let \( g = \mathcal{C}_{p,r}(D) \). Set
\[
h(z) = b^\sigma(z,z) \Lambda_p g(z) \quad \text{and} \quad f = \mathcal{P}_r h ;
\]
then by Theorem VI.1, \( f \) belongs to \( A^{p,r}(D) \) under our assumptions on \( p \), since \( h \) belongs to \( L^{p,r}(D) \). Therefore \( \Lambda_p f = \mathcal{P}_r (\Lambda_p g) \). On the other hand, \( \Lambda_p g \in A^{p-sigma}(D) \); then \( \mathcal{P}_r (\Lambda_p g) = \Lambda_p g \). Hence \( \Lambda_p g = \Lambda_p f \) and then, \( f \) is a representative of the class \( g \) in \( \mathcal{C}_{p,r}(D) \).

Next, let \( f \in A^{p,r}(D) \) be such that \( \Lambda_p f = 0 \). Then \( f = \mathcal{P}_r f \), and this implies that
\[
\mathcal{T}_{r+\sigma, r} f(z) = b^{-\sigma}(z, z) \Lambda_p f(z) = 0.
\]
By the Fubini theorem, one easily obtains that
\[
P_t \left[ T_{r+\sigma, r} f \right] (\zeta)
= c_\mathcal{D} f(\eta)b^{-r -(\eta, \eta)} \left( \int _\mathcal{D} b^{1+r + \sigma (z, \eta)} b^{1+r - \sigma (z, z)} dv(z) \right) dv(\eta)
= c_\mathcal{D} f(z) b^{-r -(\eta, \eta)} b^{1+r (\zeta, \zeta)} dv(\eta),
\]
where the latest equality follows from the reproducing formula \( P_{r+\sigma} (b^{1+r} (\cdot, \cdot)) (\eta) = b^{1+r} (\eta, \zeta) \) for all \( \eta \) and \( \zeta \) in \( D \), since the function \( b^{1+r} (\cdot, \cdot) \) belongs to \( A^{p,r}(D) \) and \( \rho_i > \frac{1}{2} b_i \) for all \( i = 1,...,l \). Finally, we conclude that \( P_t (\mathcal{T}_{r+\sigma, r} f) = P_t (f) = f \), and since \( \mathcal{T}_{r+\sigma, r} f = 0 \), this implies that \( f = 0 \). Hence each class in \( \mathcal{C}_{p,r}(D) \) has only one representative in \( A^{p,r}(D) \).
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Conversely, let \( f \in A^p_r(D) \); then \( P_r f = f \) and \( A^p_r f(\zeta) \sigma = T_{r+\sigma} f(\zeta) \), for all \( \zeta \) in \( D \). Hence, by Theorem VI.1, \( T_{r+\sigma} f \) belongs to \( L^p_r(D) \) since

\[
p > \max_i \left\{ \frac{2n_i + 2d - 2d_q r_i}{n_i + 2d - 2d_q r_i} \right\} \quad \text{and} \quad \rho_i > \frac{n_i}{2} \quad \text{for all} \quad i = 1, \ldots, l.
\]

Therefore \( f \) is a representative of a class in \( C^p_r(D) \).

APPENDIX

PROOF OF LEMMA V. 3.

We first remark that for all \((z, u) \in D_2\):

\[(*) \quad \left| u_2 \right|^2 < \left| \zeta_2 \left( \frac{ie - z^*}{2i} \right) \right|.
\]

By this remark \((*)\), we have the estimate:

\[
\left| u_2 \right|^4 \left| \zeta_1^{-(5+r)} \zeta_2^{-8+r} \left( \frac{ie - z^*}{2i} \right) \right| \leq \left| \zeta_1^{-(5+r)} \zeta_2^{-6+r} \left( \frac{ie - z^*}{2i} \right) \right|
\]

This latest function is integrable by Theorem II.2. Hence \((a)\) is proved. The same estimate permits to conclude that \((b)\) is true. For the next three integrals, by the previous remark \((*)\), it is sufficient to prove that for a suitable choice of \( \delta \) (\( \delta = 0 \) for \((c)\) and \((d)\) and \( \delta = 0.5 \) for \((e)\)), the following function is integrable in \( D_2 \):

\[
g_\delta(z, u) = \overline{z}_1 \zeta_1^{-(5+r)} \zeta_2^{-7+r+\delta} \left( \frac{ie - z^*}{2i} \right).
\]
To this aim, we shall use computational techniques due to Békollé in [B2]. Let $\alpha_1$ and $\alpha_2$ be two real numbers such that $0 < \alpha_1 < 1$ and $1 < \alpha_2 < 3 + 2\delta$, and consider the sets $D_3$ and $D_4$ defined by:

$$D_3 = \{(z, u) \in D_2 : \|g_\delta(z, u)\| \leq \chi_1^{-1}(4 + r + \alpha_1) / \chi_2^{-1}(4 + r + \alpha_2) \left( \frac{ie - z^*}{2i} \right) \}$$

$$D_4 = D_2 \setminus D_3.$$

In $D_1$, $|g_\delta(z, u)|$ is dominated by an integrable function in $D_2$ by our assumption on $\alpha_1$ and $\alpha_2$ and Theorem II. 2. In $D_4$, it is obvious that

$$|g_\delta(z, u)| \leq c|h_\delta(z, u)|^2,$$

where $h_\delta(z, u)$ is equal to:

$$h_\delta(z, u) = \chi_1^{-1} \chi_2^{-1} \left( \frac{6 + r - \alpha_1}{2}, \frac{10 - \alpha_2 + r + 2\delta}{2} \right) \left( \frac{ie - z^*}{2i} \right).$$

From the very definition of $\chi_1$ and $\chi_2$, we have:

$$h_\delta(z, u) = c \frac{\partial}{\partial z} \left\{ \frac{-(r - \alpha_1 + 4) / 2, -(8 - \alpha_2 + r + 2\delta) / 2 \left( \frac{ie - z^*}{2i} \right) }{\chi_1^{-1} \chi_2^{-1}} \right\}.$$

Therefore one easily gets:

$$h_\delta(z, u) = c \exp \left\{ - \frac{ie - z^*}{2i} \right\} \chi_1^{-1} \chi_2^{-1} \left( \frac{4 - \alpha_2 + r + 2\delta}{2}, \frac{r - \alpha_1}{2}, \frac{r}{\lambda} \right),$$

where $\chi_1(\lambda) = \lambda_2^{1/2} - \lambda_1^{1/2}$ and $\chi_2(\lambda) = \lambda_1^{1/2}$ are two defining functions of $V = V^*$. Hence, by the Plancherel-Gindikin formula [BT] with $C^m = M_{r^2}$, we get:
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\[
\int_{D_2} |h_\delta(z,u)|^2 \, dv(z,u) = c_0 \int_{\mathcal{C}} m \, dv(u)
\]

\[
\left( \int_{\mathcal{C}} \exp(-\lambda \cdot e) |h_{12}|^2 \chi_1^* \chi_2 \right)^2 \chi_2 \exp(2 \delta \chi_2) \, \exp(-2 \lambda + \frac{r}{2} - \frac{\alpha_1 + \frac{r}{2}}{2} \exp(-2 \lambda + u \cdot u > d\lambda) \right)
\]

\[
= c_0 \int_{\mathcal{C}} \exp(-\lambda_{11} + \lambda_{22} \lambda_{11} \lambda_{22} - 4 \left| \lambda_{12} \right|^2) \chi_1^{*} \chi_2^{*} \lambda_{12}^{*} \, d\lambda_{12}^{*} \chi_2^{*} \chi_1^{*}
\]

By a polar coordinate change of variables, the integration with respect to \( \lambda_{12} \)
converges if \( \alpha_2 < 3 + 2\delta \), and we have :

\[
\int_{D_2} |h_\delta(z,u)|^2 \, dv(z,u) = \int_0^{\infty} \int_0^{\infty} \exp(-\lambda_{11} + \lambda_{22} \lambda_{11} \lambda_{22} - 4 \left| \lambda_{12} \right|^2) \chi_1^{*} \chi_2^{*} \lambda_{12}^{*} \, d\lambda_{12}^{*} \chi_2^{*} \chi_1^{*}
\]

Henceforth, this integral converges by our assumption on \( \alpha_1 \) and \( \alpha_2 \). This proves
assertions (c), (d) and (e).

Following the same pattern, it is sufficient to prove that the function \( f_\delta(z,u) \)
defined by :

\[
f_\delta(z,u) = \left| u_1 \right| \chi_1^{(5+r)} \chi_2^{-(6+r+\delta)} \left( \frac{\text{ie} - z^*}{2\text{i}} \right)
\]
is integrable in \( D_2 \) ( for (f), take \( \delta = 1 \); for (g), take \( \delta = 0.5 \), and for (h), take \( \delta = 0 \) ).

The techniques used before imply that it is sufficient to prove that if \( 0 < \alpha_1 < 1 \) and \( 1 < \alpha_2 < 2 + 2\delta \), we have :

\[
|f_\delta(z,u)| \leq |K_\delta(z,u)|^2
\]

where \( K_\delta(z,u) = \left| u_1 \right| \chi_1 \chi_2 \left( \frac{6 - \alpha_1 + r}{2} \chi_1 \chi_2 - \frac{8 - \alpha_2 + r + 2\delta}{2} \left( \frac{\text{ie} - z^*}{2\text{i}} \right) \right) \) is square integrable
in \( D_2 \). It is clear that :
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\[ K_\delta(z,u) = c |u_1| |V \exp(- \frac{\lambda}{2} \chi_1^* \frac{2 - \alpha_1 + r}{2}) \]  

then:

\[ \int_{D^2} |K_\delta(z,u)|^2 d\nu(z,u) \]

\[ = c \int |\exp(- \lambda \chi_1^* \frac{2 - \alpha_2 + r + 2\delta}{\lambda}) \chi_2^* \frac{2 - \alpha_1 + r}{2}) \lambda d\lambda d\nu(u) \]

\[ = c \int |\exp(- (\lambda_1^2 + \lambda_2^2) \frac{2 - \alpha_2 + 2\delta - \alpha_1^2 - \alpha_1 \lambda_2^2}{2}) \lambda d\lambda d\nu(u) \]

\[ = c \int_{[1, \infty)} |\exp(- (\lambda_1^2 + \lambda_2^2) \frac{2 - \alpha_2 + 2\delta - \alpha_1 \lambda_2^2}{2}) \lambda d\lambda, \]

This converges by our assumption on \( \alpha_1 \) and \( \alpha_2 \). □

**PROOF OF THEOREM II.3.**

We first prove the following lemma:

A **LEMMA.** Let \( p \in [1, \infty) \) and let \( r \) be a vector of \( \mathbb{R}^l \) such that \( r_i \geq -1 \) (\( i = 1, \ldots, l \)). Let \( f \in A^{p,r}(D) \). Then for every \((y,u) \in V \times C^m\) such that \( y - F(u,u) \in V \), the holomorphic function \( f_{y,u} \) defined on \( \mathbb{R}^n + iV \) by

\[ f_{y,u}(z) = f(z+iy,u) \]

belongs to the Hardy space \( H^p(\mathbb{R}^n + iV) \). Moreover, there is a constant \( C = C(p,r) \) such that for all \( f \in A^{p,r}(D) \) and \((y,u) \in V \times C^m\) satisfying \( y - F(u,u) \in V \), then

\[ \left\| f_{y,u} \right\|_{H^p(\mathbb{R}^n + iV)} \leq C b^{\frac{1}{1/p} + r}((iy,u),(iy,u)) \| f \|_{p,r}^p. \]

**Proof of Lemma** A. First take \((y,u) = (e,0)\). Let \( P \) be a polydisc centered at \((ie,0)\) with
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closure contained in D. Let \((R_1, \ldots, R_{n+m})\) be the multiradius of \(P\). Then:

\[
(1) \quad \left| f(x + ie, 0) \right|^P \leq C \int_{\mathbb{R}^n} \left| f(\xi + i\eta, v) \right|^P d\xi d\eta d\nu(v).
\]

If \(P'\) is the projection of \(P\) on \(iV \times C^m\), we obtain:

\[
(2) \quad \int_{\mathbb{R}^n} \left| f(\xi + i\eta, v) \right|^P d\xi d\eta d\nu(v) \leq \left| \xi - x \right| < R_1 \int_{\mathbb{R}^n} \left| f(\xi + i\eta, v) \right|^P d\eta d\nu(v) d\xi.
\]

Combining (1) and (2) yields

\[
\int_{\mathbb{R}^n} \left| f(x + ie, 0) \right|^P dx \leq C (R_1)^n \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \left| f(\xi + i\eta, v) \right|^P d\eta d\nu(v) \right) d\xi.
\]

Since \(\overline{P'}\) is a compact set contained in \(\{(\eta, v) \in iV \times C^m : \eta - F(u, v) \in V\}\), there are two positive constants \(c_r\) and \(c'_r\) such that \(c'_r \leq b^{-r}(\zeta + i\eta, v), (\xi + i\eta, v) \leq c_r\) for all \((\eta, v) \in P'\) and \(\xi \in \mathbb{R}^n\). Hence, for every \(f \in H(D)\), the following holds:

\[
\int_{\mathbb{R}^n} \left| f(x + ie, 0) \right|^P dx \leq C r \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \left| f(\xi + i\eta, v) \right|^P b^{-r}(\xi + i\eta, v), (\xi + i\eta, v) d\eta d\nu(v) \right) d\xi
\]

Next assume that \((iy, u)\) is an arbitrary point of \(D\). Since \(D\) is affine-homogeneous, there is \(g \in G(D)\) such that \(g^{-1}(x+iy, u) = (x+ie, 0)\) for every \(x\) in \(\mathbb{R}^n\). Set \(h = fog\). Then

\[
\int_{\mathbb{R}^n} \left| f(x + iy, u) \right|^P dx = \int_{\mathbb{R}^n} \left| h(x + ie) \right|^P dx \leq C r \int_D \left| h(\xi + i\eta, v) \right|^P b^{-r}(\xi + i\eta, v), (\xi + i\eta, v) d\eta d\nu(v).
\]

Make the change of variables \(\xi = \xi', (i\eta, v) = g^{-1}(i\eta', v')\). Then the equality

\[
\left| \det g \right|^{-2} = cb((x + iy, u), (x + iy, u))
\]

yields that

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Furthermore, for $r \in \mathbb{R}^l$, one can prove that
\[ b(\xi + i\eta,\nu) = b(\xi' + i\eta',\nu') \det g \]
\[ = e^b((\xi' + i\eta',\nu'),(\xi + i\eta,\nu))b^{-1}((x + iy,u),(x + iy,u)). \]

Hence:
\[ \mathcal{R}_n f(y,u,p) = c_r b^{1 + r}((x + iy,u),(x + iy,u)) \|\| p_r. \]

So,
\[ \| f(y,u) \|_{L^p(\mathbb{R}^n + i\mathbb{V})} = \sup_{\eta \in \mathbb{V}} \mathcal{R}_n f(x + iy,u) \|\|_{p_r} \]
\[ \leq c_r \sup_{\eta \in \mathbb{V}} \mathcal{R}_n b^{1 + r}((x + iy,u),(x + iy,u)) \|\|_{p_r} \]
\[ = c_r b^{1 + r}((x + iy,u),(x + iy,u)) \|\|_{p_r}. \]

The last equality follows from Proposition II.4 since $r_i \geq -1$ ($i = 1,\ldots,l$).

\[ \square \]

The following corollary can be deduced from results in [SW]:

**B COROLLARY.** Let $p \in [1, \infty)$ and let $r$ be a vector of $\mathbb{R}^l$ such that $r_i \geq -1$ ($i = 1,\ldots,l$).

Let $(iy,u) \in D$ and $y' \in V$. Then for all $f \in A^{p,r}(D)$,
\[ \int_{\mathbb{R}^n} f(x + iy,y')^p dx \leq \int_{\mathbb{R}^n} f(x + iy)^p dx \]
and

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\[
\lim_{y' \to 0} \int_{\mathbb{R}^n} \left| f(x + i(y + y'), u) - f(x + iy, u) \right|^p \, dx = 0.
\]

**PROOF OF THEOREM II.3:** Let \( \varepsilon \) be a vector of \( \mathbb{R}^l \) such that \( \varepsilon_i > \frac{n_i + 2}{2(2d - q)i} \) (i = 1,...,l).

Recall that for every \( f \in A^2, \varepsilon, \alpha (D) \) , \( P_{\varepsilon} f \) is given by the following formula:

\[
P_{\varepsilon} f(z, u) = c_{\varepsilon} f_{\varepsilon} (z, u) = \frac{1}{b} \left( \frac{e - r}{p(r + 1)} \right)^{p} (z, u) \]  

\[
\int_{\mathbb{D}} b^{1+e\left((z, u), (\zeta, v)\right)} f(\zeta, v) b^{e\left((\zeta, v), (\zeta', v)\right)} dv(\zeta, v) < \infty
\]

when \( p > 1 \) (resp. if

\[
\sup_{(\zeta, v) \in D} \left\{ b^{1+e\left((z, u), (\zeta, v)\right)} b^{-e\left((\zeta, v), (\zeta', v)\right)} \right\} < \infty
\]

when \( p = 1 \) for all \((z, u) \in D\). By Theorem II.2 (resp. by Proposition II.4) , this is the case if and only if

\[
\varepsilon_i > \frac{n_i + 2}{2(2d - q)i} \]

(i = 1,...,l) and

\[
p < \min_{i=1,...,l} \left\{ \frac{n_i - 2(2d - q)i(1 + r_i)}{n_i} \right\} \text{ when } p > 1
\]

(resp. if

\[
1 + \varepsilon_i \geq \varepsilon_i - r_i \geq 0 \text{ (i = 1,...,l) when } p = 1 \].

For \( p = 1 \), this reduces to the two conditions \( r_i \geq 1 \) and \( \varepsilon_i \geq r_i \) (i = 1,...,l).

It suffices to show that for every \((\zeta, v) \in D\), under our assumptions on \( \varepsilon, r \) and \( p \), the bounded linear functional \( \varphi_{(\zeta, v)} \) defined on \( A^{p, r} (D) \) by

\[
\varphi_{(\zeta, v)}(f) = f(\zeta, v) - c_{\varepsilon} f_{\varepsilon} (\zeta, v) = \frac{1}{b} \left( \frac{e - r}{p(r + 1)} \right)^{p} (\zeta, v) \]  

\[
\int_{\mathbb{D}} b^{1+e\left((\zeta, v), (z, u)\right)} f(\zeta, v) b^{-e\left((\zeta, v), (z, u)\right)} dv(z, u)
\]

which is identically 0 on \( A^{p, r} (D) \cap A^{2, e} (D) \), vanishes identically on \( A^{p, r} (D) \). The
desired conclusion follows from the following lemma:

**Lemma C**: Let \( \varepsilon \) and \( r \) be two vectors of \( \mathbb{R}^l \) such that \( \varepsilon_i > \frac{n_i + 2}{2(2d-q)i} \) and \( r_i > \frac{n_i + 2}{2(2d-q)i} \) (\( i = 1, \ldots, l \)). Then for every \( p \in [1,\infty) \), the subspace \( A^{p,r} (D) \cap A^{2, \varepsilon} (D) \) is dense in \( A^{p,r} (D) \).

**Proof of Lemma C**. We use the following notations: for \( z = (x+iy) \in \mathbb{D} \), we write \( z_n \) for \( \frac{x}{n} - \frac{u}{\sqrt{n}} \), and we write \( i\varepsilon \) for \( (i\varepsilon, 0) \). Let \( f \in A^{p,r} (D) \). Let \( \alpha \) be a positive number to be chosen later. Consider the sequence \( \{f_q\} \) defined by

\[
f_q(z) = c_{\alpha} f\left(z + \frac{i\varepsilon}{q}\right) b^\alpha\left(\frac{z}{q}, i\varepsilon\right) \quad (z \in \mathbb{D}),
\]

where \( c_{\alpha} = b^{-\alpha}(0, i\varepsilon) \). We will show that for \( \alpha \) large, \( f_q \in A^{p,r} (D) \cap A^{2, \varepsilon} (D) \) for every positive integer \( q \), and that

\[
\lim_{q \to \infty} \|f - f_q\|_{p,r} = 0.
\]

The function \( z \mapsto f\left(z + \frac{i\varepsilon}{q}\right) \) is bounded by Lemma II.5 and Proposition II.4. Take \( \alpha \) big enough so that the function \( z \mapsto b^\alpha\left(\frac{z}{q}, i\varepsilon\right) = c_{\alpha} b^\alpha (z, i\varepsilon) \) belongs to \( A^{p,r} (D) \cap A^{2, \varepsilon} (D) \) for every \( q \in \mathbb{N} \). Next, by the Minkowski inequality, we have the estimate:

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\[
\| f_q \|_{p,r} \leq \left[ \int_D |f(z)|^p \left| c_{\alpha} b^{\alpha}(\frac{z}{q}, i \epsilon) - 1 \right|^p dv(z) \right]^{\frac{1}{p}}
\]

\[
+ \left[ \int_D \left| c_{\alpha} b^{\alpha}(\frac{z}{q}, i \epsilon) \right|^p \left| f \left( z + \frac{i \epsilon}{q} \right) - f(z) \right|^p b^{-r}(z,z) dv(z) \right]^{\frac{1}{p}}.
\]

By Lemma II.4, there is a positive constant \( A_\alpha \) such that for all \( z \in D \) and \( q \in \mathbb{N} \), we have

\[
\left| c_{\alpha} b^{\alpha}(\frac{z}{q}, i \epsilon) \right| \leq A_\alpha.
\]

Hence, by the dominated convergence theorem, the first integral on the right side of the estimate goes to 0 as \( q \) goes to infinity. Now, to study the second integral, it suffices to prove that

\[
I_q = \int_D \left| f \left( z + \frac{i \epsilon}{q} \right) - f(z) \right|^p b^{-r}(z,z) dv(z)
\]

goes to 0 when \( q \) goes to infinity. Set \( z = (x+iy, u) \in D \) and obtain that \( I_q \) is equal to

\[
\int_{\mathbb{R}^n} \int_{V} F(u, u) \left( \int_{\mathbb{R}^n} \left| f \left( x + iy \left( 1 + \frac{e}{q} \right), u \right) - f(x + iy, u) \right|^p \right) \left( y - F(u, u) \right)^{- (2d - q)r} dy \right] dv(u).
\]

Observe that Corollary B yields the following two facts:

\[
\int_{\mathbb{R}^n} \left| f \left( x + iy \left( 1 + \frac{e}{q} \right), u \right) - f(x + iy, u) \right|^p \right) dx \leq 2^p \int_{\mathbb{R}^n} \left| f(x + iy) \right|^p dx
\]

and

\[
\lim_{q \to \infty} \int_{\mathbb{R}^n} \left| f \left( x + iy \left( 1 + \frac{e}{q} \right), u \right) - f(x + iy, u) \right|^p \right) dx = 0.
\]

On the other hand, since \( f \in A^{p, r}(D) \), the function \((y, u) \mapsto \int_{\mathbb{R}^n} \left| f(x + iy, u) \right|^p dx\)
is integrable on \( \{(y,u) : y \in V+F(u,u), u \in C^m \} \) with respect to the measure 
\((y-F(u,u))^{-(2d-q)r} \, dy \, dv(u) \). Hence, by the dominated convergence theorem, it follows that 
\( I_q \) goes to 0 as \( q \) tends to infinity. Lemma C is proved.

Thus Theorem II.3 is entirely proved.

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Department of Mathematics
Faculty of Science P.O. Box 812
University of Yaounde I
Yaounde (Cameroon)