Uniqueness of generators of strong symplectic isotopies

Augustin Banyaga  
Department of Mathematics, The Pennsylvania State University.  
University Park, PA 16802. United States.  
banyaga@math.psu.edu

Stéphane Tchuiaga  
Department of Mathematics, University of Buea, South West Region, PO Box 63, Cameroon.  
tchuiagas@gmail.com

Abstract

The group $SSympeo(M,\omega)$ of strong symplectic homeomorphisms of any closed symplectic manifold $(M,\omega)$ was defined and studied in [4] and further in [6]. In this paper, we prove that a continuous symplectic flow (or a strong symplectic isotopy) admits a unique generator. This generalizes Viterbo's result in the Hamiltonian case [15].
Uniqueness of generators of strong symplectic isotopies

Augustin Banyaga and Stéphane Tchuiaga

Abstract. The group $SSympeo(M, \omega)$ of strong symplectic homeomorphisms of any closed symplectic manifold $(M, \omega)$ was defined and studied in [4] and further in [6]. In this paper, we prove that a continuous symplectic flow (or a strong symplectic isotopy) admits a unique generator. This generalizes Viterbo’s result in the Hamiltonian case [15].


Keywords. Hofer-like norm, Hodge theory, Homotopy, Isotopy, Symplectomorphism, symplectic homeomorphisms, mass flows, flux, Hopf-Rinow theorem, injectivity radius.

1. Introduction

The study of continuous symplectic geometry began with the celebrated Eliashberg-Gromov rigidity theorem, which states that the group of symplectomorphisms of any symplectic manifold $(M, \omega)$ is $C^0$ closed in the group of diffeomorphisms of the manifold $M$. This theorem has motivated the construction of various groups of symplectic homeomorphisms.

A continuous family $(\gamma_t)_{t \in [0, 1]}$ of homeomorphism of $M$ with $\gamma_0 = id$ is called a continuous symplectic flow (or strong symplectic isotopy ) if there exists a sequence $\Phi_i = (\phi^i_t)_{t \in [0, 1]}$ of symplectic isotopies which converges uniformly to $(\gamma_t)_{t \in [0, 1]}$ and such that the sequence of derivatives $(\dot{\phi}^i_t)$ with respect to the time variable is Cauchy for the $L^\infty$ norm (see [3]).

The goal of this paper is to show (Theorem 3.7), that the limit

$$\mathcal{M}(\Phi_i) := \lim_{L^\infty}((\dot{\phi}^i_t)),$$

is independent of the choice of the sequence of symplectic isotopies $\Phi_i = (\phi^i_t)_{t \in [0, 1]}$; we call it a "generator" of the strong symplectic isotopy $(\gamma_t)_{t \in [0, 1]}$. This generalizes the uniqueness theorem of generating functions for continuous Hamiltonian flows of Viterbo [15], Buhovsky-Seyfaddini [7].

2. Preliminaries

Let $M$ be a smooth closed manifold of dimension $2n$. A differential 2–form $\omega$ on $M$ is called a symplectic form if $\omega$ is closed and nondegenerate. In particular, any symplectic manifold

The second author thanks the German Academic Exchange Service (DAAD) for the financial support of his research at the Institute of Mathematics and Physical Sciences (IMSP).
is oriented. From now on, we shall always assume that \( M \) admits a symplectic form \( \omega \). A diffeomorphism \( \phi : M \to M \) is called symplectic if it preserves the symplectic form \( \omega \), i.e. \( \phi^*(\omega) = \omega \). We denote by \( \text{Symp}(M, \omega) \) the symplectomorphisms’ group.

### 2.1. Symplectic vector fields

The symplectic structure \( \omega \) on \( M \), being nondegenerate, induces an isomorphism between vector fields \( Z \) and 1–forms on \( M \) given by \( Z \mapsto \omega(Z, \cdot) = \iota(Z)\omega \). A vector field \( Z \) on \( M \) is said to be a Hamiltonian vector field if \( \iota(Z)\omega \) is exact. It follows from the definition of symplectic vector fields that, if the first de Rham cohomology group of the manifold \( M \) is trivial (i.e. \( H^1(M, \mathbb{R}) = 0 \)), then all the symplectic vector fields induced by a symplectic form \( \omega \) on \( M \) are Hamiltonian. If we equip \( M \) with a Riemannian metric \( g \), then any harmonic 1–form \( \alpha \) on \( M \) determines a symplectic vector field \( Z \) such that \( \iota(Z)\omega = \alpha \) (see [3]).

### 2.2. Symplectic isotopies

An isotopy \( (\phi_t)_{t \in [0,1]} \) of the symplectic manifold \( (M, \omega) \) is said to be symplectic if for each \( t \in [0,1] \), the vector field \( Z_t = \frac{d\phi_t}{dt} \circ \phi_t^{-1} \) is symplectic. In fact, a symplectic isotopy \( (\phi_t)_{t \in [0,1]} \) is a smooth family of symplectic diffeomorphisms such that \( \phi_0 = \text{Id}_M \). In particular a symplectic isotopy \( (\psi_t)_{t \in [0,1]} \) is called Hamiltonian if for each \( t \), \( Z_t = \frac{d\psi_t}{dt} \circ \psi_t^{-1} \) is Hamiltonian, i.e. there exists a smooth function \( F : [0,1] \times M \to \mathbb{R} \) called Hamiltonian such that \( \iota(Z_t)\omega = dF_t \). As we can see, any Hamiltonian isotopy determines a Hamiltonian \( F : [0,1] \times M \to \mathbb{R} \) up to an additive constant. Throughout the paper we assume that all Hamiltonians are normalized in the following way: given a Hamiltonian \( F : [0,1] \times M \to \mathbb{R} \) we require that \( \int_M F_t \omega^n = 0 \) for all \( t \). We denote by \( \mathcal{N}([0,1] \times M, \mathbb{R}) \) the space of all smooth normalized Hamiltonians and by \( \text{Ham}(M, \omega) \) the set of all time-one maps of Hamiltonian isotopies. If we equip \( M \) with a Riemannian metric \( g \), then a symplectic isotopy \( (\theta_t)_{t \in [0,1]} \) is said to be harmonic if for each \( t \), \( Z_t = \frac{d\theta_t}{dt} \circ \theta_t^{-1} \) is harmonic. We denote by \( \text{Iso}(M, \omega) \) the group of all symplectic isotopies of \( (M, \omega) \) and by \( \text{Symp}_0(M, \omega) \) the set of all time-one maps of symplectic isotopies.

### 2.3. Harmonics 1–forms [13]

From now on, we assume that \( M \) is equipped with a Riemannian metric \( g \), and denote by \( \mathcal{H}^1(M, g) \) the space of harmonic 1–forms on \( M \) with respect to the Riemannian metric \( g \). In view of the Hodge’s theory, \( \mathcal{H}^1(M, g) \) is a finite dimensional vector space over \( \mathbb{R} \) which is isomorphic to \( H^1(M, \mathbb{R}) \) (see [16]). The dimension of \( \mathcal{H}^1(M, g) \) is the first Betti number of the manifold \( M \), denoted by \( b_1 \). Taking \( (h^1_{\lambda})_{1 \leq \lambda \leq b_1} \) as a basis of the vector space \( \mathcal{H}^1(M, g) \), we equip \( \mathcal{H}^1(M, g) \) with the Euclidean norm \( \| \cdot \| \) defined as follows: for all \( H \) in \( \mathcal{H}^1(M, g) \) with \( H = \sum_{1 \leq \lambda \leq b_1} \lambda^i h^i \) we have \( |H| := \Sigma_{i=1}^{b_1} |\lambda^i| \). It is convenient to compare the above Euclidean norm with the well-known uniform sup norm of differential 1–forms. For this purpose, let’s recall the definition of the uniform sup norm of a differential 1–form \( \alpha \) on \( M \). For all \( x \in M \), we know that \( \alpha \) induces a linear map \( \alpha_x : T_x M \to \mathbb{R} \) whose norm is given by \( \| \alpha_x \| = \sup \{ |\alpha_x(X)| : X \in T_x M, \|X\|_g = 1 \} \), where \( \| \cdot \|_g \) is the norm induced on each tangent space \( T_x M \) by the Riemannian metric \( g \). Therefore, the uniform sup norm of \( \alpha \), say \( |\alpha|_0 \) is defined by \( |\alpha|_0 = \sup_{x \in M} \| \alpha_x \| \).

In particular, when \( \alpha \) is a harmonic 1–form (i.e \( \alpha = \Sigma_{i=1}^{b_1} \lambda^i h^i \)), we obtain the following estimates:

\[
|\alpha|_0 \leq \Sigma_{i=1}^{b_1} |\lambda^i| |h^i|_0 \leq E |\alpha|,
\]

Imhotep J.
where

\[ E := \max_{1 \leq i \leq b_1} |h^i|_0. \]

If the basis \((h^i)_{1 \leq i \leq b_1}\) is such that \(E > 1\), then one can always normalize such a basis so that \(E\) equals 1. Otherwise, the identity \(|\alpha|_0 \leq E|\alpha|\) reduces to \(|\alpha|_0 \leq |\alpha|\). We denote by \(\mathcal{P}\mathcal{H}^1(M, g)\), the space of smooth mappings \(\mathcal{H} : [0, 1] \to \mathcal{H}^1(M, g)\).

2.4. A description of symplectic isotopies [6]

In this subsection, from the group of symplectic isotopies, we shall deduce another group which will be convenient later on (see [6]). Consider \((\phi_t)_{t \in [0, 1]}\) to be a symplectic isotopy, for each \(t\), the vector field \(Z_t = \frac{d\phi_t}{dt} \circ (\phi_t)^{-1}\) satisfies \(d(Z_t)\omega = 0\). So, it follows from Hodge’s theory that \(\iota(Z_t)\omega\) decomposes as the sum of an exact 1–form \(dU^\phi_t\) and a harmonic 1–form \(\mathcal{H}^\phi_t\) (see [16]). Denote by \(U\) the Hamiltonian \(U^\phi = (U^\phi_t)\) normalized, and by \(\mathcal{H}\) the smooth family of harmonic 1–forms \(\mathcal{H}^\phi = (\mathcal{H}^\phi_t)\). In [6], the authors denoted by \(\Xi(M, \omega, g)\) the Cartesian product \(\mathcal{N}([0, 1] \times M, \mathbb{R}) \times \mathcal{P}\mathcal{H}^1(M, g)\), and equipped it with a group structure which makes the bijection

\[ Iso(M, \omega) \to \Xi(M, \omega, g), \Phi \mapsto (U, \mathcal{H}) \quad (2.1) \]

a group isomorphism. Denoting the map just constructed by \(\mathfrak{A}\), the authors denoted any symplectic isotopy \((\phi_t)_{t \in [0, 1]}\) as \(\phi(U, \mathcal{H})\) to mean that the mapping \(\mathfrak{A}\) maps \((\phi_t)_{t \in [0, 1]}\) onto \((U, \mathcal{H})\), and \((U, \mathcal{H})\) is called the "generator" of the symplectic path \(\phi(U, \mathcal{H})\). In particular, any symplectic isotopy of the form \(\phi(0, \mathcal{H})\) is considered to be a harmonic isotopy, while any symplectic isotopy of the form \(\phi(U, 0)\) is considered to be a Hamiltonian isotopy. The product in \(\Xi(M, \omega, g)\) is given by,

\[ (U, \mathcal{H}) \otimes (V, \mathcal{K}) = (U + V \circ \phi^{-1}_0(U, \mathcal{H}) + \tilde{\Delta}(\mathcal{K}, \phi^{-1}_0(U, \mathcal{H})), \mathcal{H} + \mathcal{K}) \quad (2.2) \]

The inverse of \((U, \mathcal{H})\), denoted \((U, \mathcal{H})^{-1}\) is given by

\[ (U, \mathcal{H})^{-1} = (-U \circ \phi(U, \mathcal{H}) - \tilde{\Delta}(\mathcal{H}, \phi(U, \mathcal{H})), -\mathcal{H}) \quad (2.3) \]

where for each \(t\), \(\phi^{-1}_t(U, \mathcal{H}) := (\phi^t(U, \mathcal{H}))^{-1}\), and \(\tilde{\Delta}_t(\mathcal{K}, \phi^{-1}_t(U, \mathcal{H}))\) is the function

\[ \Delta_t(\mathcal{K}, \phi^{-1}_t(U, \mathcal{H})) := \int_0^1 \kappa_t(\phi^{-s}_t(U, \mathcal{H}) \circ \phi^{-1}_t(U, \mathcal{H})^{-s}) ds \quad \text{normalized}. \]

2.5. The metric space \(\Xi(M, \omega, g)\) [3, 6]

In the style of the first author in [3, 4], let us define the following metric. For all \((U, \mathcal{H}), (V, \mathcal{K}) \in \Xi(M, \omega, g)\), we define the \(L^\infty\) distance between them by the following formula :

\[ D^2((U, \mathcal{H}), (V, \mathcal{K})) = \frac{D_0^\infty((U, \mathcal{H}), (V, \mathcal{K})) + D_0^\infty((U, \mathcal{H}), (V, \mathcal{K}))}{2}(L^\infty - \text{topology}) \quad (2.4) \]

where,

\[ D_0^\infty(U, \mathcal{H}), (V, \mathcal{K}) = \max_{t \in [0, 1]} [\text{osc}(U_t - V_t) + |\mathcal{H}_t - \mathcal{K}_t|] \quad (2.5) \]

and

\[ \text{osc}(f) = \max_{x \in M} f(x) - \min_{x \in M} f(x), \quad (2.6) \]

for all continuous function \(f : M \to \mathbb{R}\).

Imhotep J.
2.6. The $C^0$–metric

Let $Homeo(M)$ be the homeomorphisms’ group of $M$ equipped with the $C^0$–compact-open topology. This is the metric topology induced by the distance

$$d_0(f,h) = \max(d_{C^0}(f,h),d_{C^0}(f^{-1},h^{-1}))$$

where $d_{C^0}(f,h) = \sup_{x \in M}d(h(x),f(x))$ and $d$ is a distance on $M$ induced by the Riemannian metric $g$. On the space of all continuous paths $g : [0,1] \to Homeo(M)$ such that $g(0) = id$, we consider the $C^0$–topology as the metric topology induced by the distance $d(\lambda,\mu) = \max_{t \in [0,1]}d_0(\lambda(t),\mu(t))$.

2.7. The flux (Banyaga) [1, 2]

Denote by $Diff(M)$ the group of diffeomorphisms on $M$ endowed with the $C^\infty$–compact-open topology. Let $Diff_0(M)$ be the identity component of $Diff(M)$ for the $C^\infty$ compact open topology. Let $\Omega$ be any closed $p$–form on $M$. Denote by $Diff^\Omega(M) \subset Diff(M)$ the space of all diffeomorphisms that preserve the $p$–form $\Omega$, and by $Diff^\Omega_0(M)$ we denote the connected component by smooth arcs of the identity in $Diff^\Omega(M)$. Let $\Phi = (\phi_t)$ be a smooth path in $Diff^\Omega_0(M)$ with $\phi_0 = Id$, and let

$$Z_t(x) = \frac{d\phi_t}{dt}((\phi_t)^{-1}(x))$$

for all $(t,x) \in [0,1] \times M$. It was proved in [2] that

$$\sum_0(\Phi) = \int_0^1(\phi_t^*\Omega(t))dt,$$

is a closed $p - 1$ form and its cohomology class denoted by $[\sum_0(\Phi)] \in H^{(p-1)}(M,\mathbb{R})$ depends only on the homotopy class $\{\Phi\}$ of the isotopy $\Phi$ relatively to fixed ends in $Diff^\Omega_0(M)$, and the map $\{\Phi\} \mapsto [\sum_0(\Phi)]$ is a surjective group homomorphism

$$\widetilde{\mathcal{S}} : Diff^\Omega_0(M) \to H^{(p-1)}(M,\mathbb{R}).$$

In case $\Omega$ is a symplectic form $\omega$, we get a homomorphism

$$\mathcal{C}al : Symp^\omega_0(M,\omega) \to H^1(M,\mathbb{R}),$$

where $Symp^\omega_0(M,\omega)$ is the universal covering of the space $Symp^\omega_0(M,\omega)$. Denote by $\Gamma$ the image under $\mathcal{C}al$ of $\pi_1(Symp^\omega_0(M,\omega))$. Due to Ono [11], $\Gamma$ is a discrete group. The homomorphism $\widetilde{\mathcal{C}}al$ induces an epimorphism $\mathcal{C}al$ from $Symp^\omega_0(M,\omega)$ onto $H^1(M,\mathbb{R})/\Gamma$. Banyaga [2, 1] proved that the group of all Hamiltonian diffeomorphisms of any compact symplectic manifold $(M,\omega)$ is a simple group which coincides with the kernel of $\mathcal{C}al$, a very deep result.

A formula which can be found in [2] shows that when $\Omega$ is the symplectic volume, then the following factorization holds:

$$\widetilde{\mathcal{S}}(\Phi) = \mathcal{C}al(\Phi) \wedge \frac{\omega^{n-1}}{(n-1)!},$$

for all symplectic isotopy $\Phi$.

2.8. The mass flow (Fathi) [8]

Let $\mu$ be a “good measure” on the manifold $M$. Let $Homeo_0(M,\mu)$ denotes the identity component in the group of measure preserving homeomorphisms $Homeo(M,\mu)$, and $Homeo_0(M,\mu)$ its universal covering. For $[h] = [(h_t)] \in Homeo_0(M,\mu)$, and a continuous $f : M \to S^1$ map, we lift the homotopy $fh_t - f : M \to S^1$ to a map $\overline{fh_t - f}$ from $M$ onto $\mathbb{R}$. Fathi proved that the
integral $\int_M H_t - f d\mu$ depends only on the homotopy class $[h]$ of $(h_t)$ and the homotopy class \{f\} of $f$ in $[M, S^1] \approx H^1(M, \mathbb{Z})$, and that the map

$$\tilde{\mathfrak{h}}((h_t))(f) := \int_M H_t - f d\mu$$

defines a homomorphism $\tilde{\mathfrak{h}} : Homeo_0(M, \mu) \to H_1(M, \mathbb{R})$. This map induces a surjective group morphism $\mathfrak{F}$ from $Homeo_0(M, \mu)$ onto a quotient of $H_1(M, \mathbb{R})$ by a discrete subgroup. This map is called the Fathi mass flow. We shall use the following Fathi’s duality result (see Proposition A.5.1 found in [8]).

**Proposition 2.1.** (Fathi, [8]) Let $\Phi$ be an isotopy of a compact oriented manifold $M$. If necessary, assume that the volume form $\Omega$ on $M$ is normalized such that $\int_M \Omega = 1$. Then the cohomology class $\tilde{\mathfrak{S}}(\Phi)$ is the Poincaré dual of the homology class $\tilde{\mathfrak{h}}(\Phi)$.

From Relation 2.9 we see that on a Lefschetz symplectic manifold, (i.e. the mapping

$$\omega^{n-1} : H^1(M, \mathbb{R}) \to H^{2n-1}(M, \mathbb{R}), \alpha \mapsto \alpha \wedge \omega^{n-1},$$

is an isomorphism) for a given symplectic isotopy $\Phi$, we have $\tilde{\mathfrak{S}}(\Phi)$ nontrivial if and only if $Cal(\Phi)$ is nontrivial. This together with Proposition 2.1 suggest that on such a manifold the mass flow of an isotopy is nontrivial if and only if its first Calabi invariant is nontrivial. Note that Lefschetz manifolds includes all Kähler manifolds, such has oriented surfaces and even dimensional tori.

### 3. Continuous symplectic flows

Let $\mathcal{N}^0([0, 1] \times M, \mathbb{R})$ be the completion of the metric space $\mathcal{N}([0, 1] \times M, \mathbb{R})$ for the $L^\infty$–Hofer norm, and $\mathcal{P}H^1(M, g)_0$ be the completion of the metric space $\mathcal{P}H^1(M, g)$ with respect to the uniform sup norm. Put,

$$J^0(M, \omega, g) := \mathcal{N}^0([0, 1] \times M, \mathbb{R}) \times \mathcal{P}H^1(M, g)_0,$$

and consider the following inclusion map $i_0 : \mathcal{F}(M, \omega, g) \to J^0(M, \omega, g)$. The map $i_0$ is uniformly continuous with respect of the topology induced by the metric $D^2$ on the space $\mathcal{F}(M, \omega, g)$, and the natural topology of the complete metric space $J^0(M, \omega, g)$. Now, let $L(M, \omega, g)$ denotes the space image$(i_0)$, and $\mathcal{F}(M, \omega, g)_0$ be the closure of the image $L(M, \omega, g)$ inside the complete metric space $J^0(M, \omega, g)$. That is, $\mathcal{F}(M, \omega, g)_0$ consists of pairs $(U, \mathcal{H})$ where the maps $(t, x) \mapsto U_t(x)$ and $t \mapsto \mathcal{H}_t$ are continuous, and for each $t$, $\mathcal{H}_t$ lies in $\mathcal{F}(M, \omega, g)$ such that there exists a $D^2$–Cauchy sequence $(U^i, \mathcal{H}^i) \subset \mathcal{F}(M, \omega, g)$ that converges to $(U, \mathcal{H})$ in $J^0(M, \omega, g)$.

**3.1. Symplectic homeomorphisms**

The following definitions can be found in [4, 6, 14].

**Definition 3.1.** A homeomorphism $h$ is called a strong symplectic homeomorphism in the $L^\infty$–context if there exists a $D^2$–Cauchy sequence $(V^i, \mathcal{K}^i) \subset \mathcal{F}(M, \omega, g)$ such that the sequence of isotopies $\phi_{V^i, \mathcal{K}^i}$ is Cauchy in $d$, and the sequence of time-one maps $(\phi_{V^i, \mathcal{K}^i}(1))$ converges uniformly to $h$.

In [6], we have proved that the set of all $L^\infty$–strong symplectic homeomorphisms coincides with the set of all strong symplectic homeomorphisms in the $L^1(1, \infty)$ topology. We denote both sets by $SS\text{Sym}eo(M, \omega)$. In this paper we will use exclusively the $L^\infty$ topology.

**Definition 3.2.** A continuous map $\xi : [0, 1] \to Homeo(M)$ with $\xi(0) = id$ will be called continuous symplectic flow if there exists a $D^2$–Cauchy sequence $(F^i, \lambda^i) \subset \mathcal{F}(M, \omega, g)$ such that the sequence of symplectic isotopies $\phi_{F^i, \lambda^i}$ converges in $d$ to $\xi$. 

Imhotep J.
We denote by \( \mathcal{PSSympeo}(M, \omega) \), the space of all \( L^\infty \)-strong symplectic isotopies of \((M, \omega)\). Notice that the sequence \((F^i, \lambda^i)\) in Definition (3.2) converges necessarily in the complete metric space \( \Xi(M, \omega, g)_0 \) to an element \((U, \mathcal{H})\). Definition 3.2 is equivalent to the definition of strong symplectic isotopies that we give in the beginning of this paper, and the above pair \((U, \mathcal{H})\) represents what we called in the beginning a "generator" of the corresponding strong symplectic isotopy.

**Definition 3.3.** We define the set \( \text{GSSympeo}(M, \omega, g) \) as the space of all the pairs \((\xi, (U, \mathcal{H}))\) where \( \xi \) is an \( L^\infty \)-strong symplectic isotopy admitting \((U, \mathcal{H})\) as a "generator", i.e. there exists a sequence \((F^i, \lambda^i) \subset \Xi(M, \omega, g)\) which is Cauchy in \( D^2 \) such that \( d(\phi_{(F^i, \lambda^i)}, \xi) \to 0, i \to \infty \), and \((F^i, \lambda^i)\) converges to \((U, \mathcal{H})\) in \( \Xi(M, \omega, g)_0 \).

In the rest of this paper, for short, we will often write \((F^i, \lambda^i) \xrightarrow{L^\infty} (U, \mathcal{H})\) to mean that the sequence \((F^i, \lambda^i)\) converges to \((U, \mathcal{H})\) in the complete topological space \( J^0(M, \omega, g) \).

### 3.2. Topological groups

**Definition 3.4.** We define the symplectic topology on the space \( \text{GSSympeo}(M, \omega, g) \) as the subspace topology induced by its inclusion in the complete topological space \( \mathcal{P}(\text{Homeo}(M), \text{id}) \times \Xi(M, \omega, g)_0 \).

**Group structure of \( \text{GSSympeo}(M, \omega, g) \).** The group structure on the space \( \text{GSSympeo}(M, \omega, g) \) is defined as follows:

For all \((\xi, (F, \lambda)), (\mu, (V, \theta)) \in \text{GSSympeo}(M, \omega, g)\), their product is given by,

\[
(\xi, (F, \lambda)) \ast (\mu, (V, \theta)) = (\xi \circ \mu, (F + V \circ (\xi^{-1} + \Delta^0(\theta, \xi^{-1}), \lambda + \theta)),
\]

and the inverse of the element \((\xi, (F, \lambda))\) is given by,

\[
\overline{(\xi, (F, \lambda))} = (\xi^{-1}, (\xi^{-1} \circ (F - \Delta^0(\lambda, \xi), \vartheta)).
\]

with

\[
\Delta^0(\theta, \xi^{-1}) := \lim_{L^\infty}(\Delta(\theta^i, \phi_{(F^i, \lambda^i)}^{-1})), \quad \Delta^0(\lambda, \xi) := \lim_{L^\infty}(\Delta(\lambda^i, \phi_{(F^i, \lambda^i)})).
\]

Notice that in relations (3.1)-(3.2), \((F^i, \lambda^i)\) and \((V^i, \theta^i)\) are two sequences in \( \Xi(M, \omega, g) \) such that

\[
(F^i, \lambda^i) \xrightarrow{L^\infty} (U, \mathcal{H}) \quad (V^i, \theta^i) \xrightarrow{L^\infty} (V, \theta).
\]

**Remark 3.5.** The functions defined in relations (3.1) and (3.2) exist since one derives from Lemma 3.9 found in [13] that both sequences of functions \( \Delta(\lambda^i, \phi_{(F^i, \lambda^i)}) \) and \( \Delta(\theta^i, \phi_{(F^i, \lambda^i)}^{-1}) \) are Cauchy in the \( L^\infty \) metric, then converge in the space \( \mathcal{N}^0([0, 1] \times M, \mathbb{R}) \).

**Theorem 3.6.** \( \text{GSSympeo}(M, \omega, g) \) is a topological group.

**Proof.** We may show that the product and inversion inside \( \text{GSSympeo}(M, \omega, g) \) are continuous.

- Continuity of the inversion map.

Let \((\gamma_k, (U_k, \mathcal{H}_k))\) be a sequence of elements of \( \text{GSSympeo}(M, \omega, g) \) that converges to \((\gamma, (U, \mathcal{H})) \) in \( \text{GSSympeo}(M, \omega, g) \). By definition of \( \text{GSSympeo}(M, \omega, g) \), for any fixed integer \( k \), there exists a sequence \((V^{k,i}, \theta^{k,i}) \subset \Xi(M, \omega, g) \) such that

\[
\begin{cases}
\tilde{d}(\phi_{(V^{k,i}, \theta^{k,i})}, \gamma_k) \to 0, i \to \infty,
(V^{k,i}, \theta^{k,i}) \xrightarrow{L^\infty} (U_k, \mathcal{H}_k),
\end{cases}
\]

Imhotep J.
and there exists a sequence \((U^j, \mathcal{H}^j) \subset \mathfrak{T}(M, \omega, g)\) which does not depend on \(k\) such that
\[
\begin{align*}
&\left\{ d(\phi(U^j, \mathcal{H}^j, \gamma)) \to 0, j \to \infty, \\
&\phi(U^j, \mathcal{H}^j) \xrightarrow{L^\infty} (U, \mathcal{H}).
\end{align*}
\]

Now, consider
\[
\psi_{i,k} := \phi(V^{k,i}, \phi^k),
\]
and
\[
\phi_j := \phi(U_j, \mathcal{H}_j),
\]
for each \(j\) and \(k\). We want to prove that the sequence \((\gamma_k^{-1}, (-U_k \circ \gamma_k - \Delta^0(\mathcal{H}_k, \gamma_k), -\mathcal{H}_k))\) converges to \((\gamma^{-1}, (-U \circ \gamma - \Delta^0(\mathcal{H}, \gamma), -\mathcal{H}))\). To do that, it suffices to show that:
\[
\begin{align*}
&\bar{d}(\gamma_k^{-1}, \gamma^{-1}) \to 0, k \to \infty, (i) \\
&\max_t \text{osc}(-U_k \circ \gamma_k + U \circ \gamma)) \to 0, k \to \infty, (\psi) \\
&\max_t \text{osc}(\Delta^0_+ (\mathcal{H}_k, \gamma_k) - \Delta^0_+ (\mathcal{H}, \gamma))) \to 0, k \to \infty, (\mathfrak{H})
\end{align*}
\]
where for each \(k\) and each \(t\), \(\Delta^0_+ (\mathcal{H}_k, \gamma_k)\) and \(\Delta^0_+ (\mathcal{H}, \gamma)\) denote respectively the limits of the sequences \(\{\int_0^t \theta_{t,i}^{-1}(\psi_{i,k})^* \phi_{i,k}^* ds\}\) and \(\{\int_0^t \mathcal{H}_i^* (\phi_j^*)^* \phi_j^* ds\}\) with respect to the uniform sup norm.

• Identity \((\psi)\). Compute
\[
\max_t \text{osc}(-U_k \circ \gamma_k - U^i \circ \gamma)) \leq \max_t \text{osc}(U_k^i \circ \gamma_k(t) - V^{k,i}_t \circ \gamma_k(t)) + \max_t \text{osc}(V^{k,i}_t \circ \gamma(t) - U_k^i \circ \gamma(t)) + \max_t \text{osc}(U^i_k \circ \gamma(t) - V^{k,i}_t \circ \gamma_k(t)) + \max_t \text{osc}(U^i_k \circ \gamma_k(t) - U^i \circ \gamma(t))
\]
for all \(i > k\). The right-hand side in the above estimate tends to zero when \(k \to \infty\), since for each \(k\), the functions \(x \mapsto V^{k,i}_t(x)\), \(x \mapsto U^i_t(x)\) and \(x \mapsto U^i_k(x)\) are uniform continuous, and \(d(\gamma_k, \gamma) \to 0, k \to \infty\), and by assumption we have the following convergences,
\[
\begin{align*}
&\max_t \text{osc}(U^i_k \circ \gamma_k(t) - U^i \circ \gamma(t)) = \max_t \text{osc}(U^i_k - U^i) \to 0, i \to \infty \\
&\max_t \text{osc}(V^{k,i}_t \circ \gamma(t) - U^i_k \circ \gamma(t)) = \max_t \text{osc}(V^{k,i}_t - U^i_k) \to 0, i \to \infty \\
&\max_t \text{osc}(U^i_k \circ \gamma_k(t) - V^{k,i}_t \circ \gamma_k(t)) = \max_t \text{osc}(U^i_k - V^{k,i}_t) \to 0, i \to \infty.
\end{align*}
\]

• Identity \((\mathfrak{H})\). For each \(t\), consider
\[
\Psi \theta^{k,i} := (\psi^i_{t,k})^* (\theta^{k,i}),
\]
and
\[
\Phi \mathcal{H}^i := (\phi_j^*)^* (\mathcal{H}_i^*).
\]
For \(l = \max\{i, j\}\), we have
\[
(\Psi \theta^{k,l} - \Phi \mathcal{H}^l) - (\theta^{k,l}_t - \mathcal{H}_i^t) = d(\mathfrak{H}_t(\theta^{k,l}, \psi_{t,k}) - \mathfrak{H}_t(\phi_j)).
\]
Now, fix a point \(m\) on \(M\), and for all \(x \in M\), pick any smooth curve \(c_x\) from \(m\) to \(x\) that minimizes the distance between \(m\) and \(x\) (such curve exists because of Hopf-Rinow theorem from Riemannian geometry, [9]), and the length of \(c_x\) is bounded from above by the diameter of the manifold \(M\), denoted here by \(\text{diam}(M)\). Consider the function
\[
u^{k,l}_t(x) = \int_{c_x} (\Psi \theta^{k,l} - \Phi \mathcal{H}^l) - (\theta^{k,l}_t - \mathcal{H}_i^t).
\]

Imhotep J.
By definition, the function $u^k_l(x)$ does not depend on the choice of the curve $c_x$, and it is easy to see that:

$$u^k_l(x) = (\Delta_t(\theta^{k,l}, \psi_{l,k}) - \Delta_t(H^l, \phi_l))(x) - ((\Delta_t(\theta^{k,l}, \psi_{l,k}) - \Delta_t(H^l, \phi_l))(m).$$

This suggests in turn that

$$\text{osc}(u^k_l) = \text{osc}(\Delta_t(\theta^{k,l}, \psi_{l,k}) - \Delta_t(H^l, \phi_l)).$$

So, to achieve the proof, we need only to show that the uniform sup norm of the function $z \mapsto |u^k_l(z)|$ tends to zero when $l>k$ and $k$ tends to infinity. For this purpose, let $y_{l,k}$ represents any point of $M$ at which the function $z \mapsto |u^k_l(z)|$ takes its maximal value.

Compute,

$$\sup_{z \in M} |u^k_l(z)| = \int_{c_{l,k}} (\Phi^{k,l} - \phi^l) = \left| \int_{c_{l,k}} (\Phi^{k,l} - \phi^l) \right|$$

and derive that

$$\left| \int_{c_{l,k}} \phi^k_l - H^l \right| \leq \left| \int_{c_{l,k}} \phi^k_l - H^l \right| + \left| \int_{c_{l,k}} H^l - \phi^l \right| + \left| \int_{c_{l,k}} \phi^l - H^l \right|$$

$$\leq (\max \phi^k_l - H^l) + \max |H^l - \phi^l| + \max |\phi^l - H^l| \text{diam}(M)$$

for all $l>k$. The right-hand side of the above estimate tends to zero when $l>k$ and $k \to \infty$, i.e.

$$\left| \int_{c_{l,k}} \theta^k_l - H^l \right| \to 0, l>k, k \to \infty.$$ 

On the other hand, from the estimate $\tilde{d}(\phi_t, \psi_{l,k}) \leq \tilde{d}(\phi_t, \gamma) + \tilde{d}(\gamma, \gamma_k) + \tilde{d}(\gamma_k, \psi_{l,k}) \to 0$, $l > k, k \to \infty$, we see that one can fix a very large integer $j_0$ and pass to subsequences so that for each $k > j_0$, the distance $\tilde{d}(\phi_t, \psi_{l,k})$ is less than $r_0$, the injectivity radius of the Riemannian metric $g$. Assume this done. Compute,

$$\left| \int_{c_{l,k}} \phi^k_l - \phi^l \right| \leq \left| \int_{c_{l,k}} \phi^k_l - \phi^l \right|$$

$$+ \left| \int_{c_{l,k}} \phi^k_l - \phi^l \right| + \left| \int_{c_{l,k}} \phi^l - \phi^l \right| + \left| \int_{c_{l,k}} \phi^l - \phi^l \right|$$

for each $t$, and all $l>k$. A straightforward calculation yields that:

$$\left| \int_{c_{l,k}} \phi^k_l - \phi^l \right| \leq \text{diam}(M) \sup_{s,t} |D\phi^l_{j_0}(c_{w_1}(s))| \max \left| \theta^k_l - H^l \right|,$$

$$\left| \int_{c_{l,k}} \phi^k_l - \phi^l \right| \leq \text{diam}(M) \sup_{s,t} |D\phi^l_{j_0}(c_{w_1}(s))| \max |H^l - H^l|,$$

$$\left| \int_{c_{l,k}} \phi^k_l - \phi^l \right| \leq \text{diam}(M) \sup_{s,t} |D\phi^l_{j_0}(c_{w_1}(s))| \max |H^l - H^l|,$$

where each right-hand side in the above estimate tends to zero when $k$ goes at infinity. On the other hand, one can always assume that $\tilde{d}(\phi_{j_0}, \phi^k_l) \leq r_0$ since

$$\tilde{d}(\phi_t, \psi_{l,k}) \leq \tilde{d}(\phi_t, \gamma) + \tilde{d}(\gamma, \gamma_k) + \tilde{d}(\gamma_k, \psi_{l,k}) \to 0, l>k, k \to \infty.$$ 

This implies that:

$$\sup_x \tilde{d}(\phi^k_l(x), \phi^l_l(x)) \leq r_0,$$
that is, $F_t$ for all $\theta$ the form

**Consider the following set,**

$$\square_{l,k} := \{ F_t(s, y_{l,k}(u)), 0 \leq s, u \leq 1 \},$$

where $\chi_{l,k}^{y_{l,k}}$ represents a minimizing geodesic that connects $y_{l,k}(z)$ to $\phi_l(z)$ (remember that $y_{l,k}$ is a point on $M$ at which the function $z \mapsto |u_{l,k}^t(z)|$ achieves its maximum). Since the form $\theta^k_{t,l}$ is closed, one derives by the help of Stokes’ theorem that

$$\int_{\partial \square_{l,k}} \theta^k_{t,l} = 0,$$

this is equivalent to

$$\int_{c_{y_{l,k}}} (\psi^l_{t,y_{l,k}})^* \theta^k_{t,l} - (\psi^l_{t,y_{l,k}})^* \theta^k_{t,l} = \int_{\chi_{l,k}^{y_{l,k}}} \theta^k_{t,l} - \int_{\chi_{l,k}^{y_{l,k}}} \theta^k_{t,l},$$

i.e. $|\int_{c_{y_{l,k}}} (\psi^l_{t,y_{l,k}})^* \theta^k_{t,l} - (\psi^l_{t,y_{l,k}})^* \theta^k_{t,l}| = \int_{\chi_{l,k}^{y_{l,k}}} \theta^k_{t,l} - \int_{\chi_{l,k}^{y_{l,k}}} \theta^k_{t,l}|. Since the speed of any geodesic is bounded from above by the distance between its end points, we derive that:

$$|\int_{\chi_{l,k}^{y_{l,k}}} \theta^k_{t,l} - \int_{\chi_{l,k}^{y_{l,k}}} \theta^k_{t,l}| \leq B_{l,k} d(\psi_{l,k}, \phi_l),$$

where

$$B_{l,k} := 2 \max |\theta^k_{t,l}|,$$

is bounded for each pair of integers $(l, k)$. That is,

$$|\int_{c_{y_{l,k}}} (\psi^l_{t,y_{l,k}})^* \theta^k_{t,l} - (\psi^l_{t,y_{l,k}})^* \theta^k_{t,l}| \to 0, l > k, k \to \infty.$$

The same arguments allow us to derive that

$$|\int_{c_{y_{l,k}}} (\psi^l_{t,y_{l,k}})^* \theta^k_{t,l} - (\psi^l_{t,y_{l,k}})^* \theta^k_{t,l}| \to 0, l > k, k \to \infty,$$

and

$$B_{l,k} := 2 \max |\theta^k_{t,l}|.$$

Finally, we have proved the identity (‴‴). For identity (‴), compute

$$d(\gamma_k^{-1}, \gamma^{-1}) = d(\gamma_k, \gamma) \to 0, k \to \infty.$$

• **Continuity of the product map.**

Let $(\gamma_k, (U_k, H_k))$ and $(\xi_k, (V_k, K_k))$ be two sequences in $GSSympeo(M, \omega, g)$ that converge respectively to $(\gamma, (U, H))$ and $(\xi, (V, K))$ in $GSSympeo(M, \omega, g)$. By characterization of an element in $GSSympeo(M, \omega, g)$, there exists two $D^2$-Cauchy sequences $(U', H')$ and $(V', K')$ in $\mathcal{E}(M, \omega, g)$ such that:

$$\left\{ \phi(U', H') \xrightarrow{C^0} \gamma \right\}$$

and

$$\left\{ \phi(V', K') \xrightarrow{C^0} \xi \right\}$$

Imhotep J.
Next, for each fixed integer $k$, there exists two sequences $(\alpha_{k,i}, \beta_{k,i})$ and $(\nu_{k,i}, \lambda_{k,i})$ such that:

$$\begin{cases}
\phi_{(\alpha_{k,i}, \beta_{k,i})} \xrightarrow{C^0} \gamma_k, \\
(\alpha_{k,i}, \beta_{k,i}) \xrightarrow{L^\infty} (U_k, \mathcal{H}_k),
\end{cases}$$

and

$$\begin{cases}
\phi_{(\nu_{k,i}, \lambda_{k,i})} \xrightarrow{C^0} \xi_k, \\
(\nu_{k,i}, \lambda_{k,i}) \xrightarrow{L^\infty} (V_k, K_k).
\end{cases}$$

Our main job is to prove that,

$$(\gamma_k, (U_k, \mathcal{H}_k)) * (\xi_k, (V_k, K_k)) \xrightarrow{C^0 + L^\infty} (\gamma, (U, \mathcal{H})) * (\xi, (V, \mathcal{K})).$$

i.e. we have to prove that $d(\gamma_k \circ \xi_k, \gamma \circ \xi) \to 0, k \to \infty$ (obvious) and

$$(U_k + V_k \circ \gamma_k^{-1} + \Delta^0(K_k, \gamma_k^{-1}), \mathcal{H}_k + K^k) \xrightarrow{L^\infty} (U + V \circ \gamma^{-1} + \Delta^0(K, \gamma^{-1}), \mathcal{H} + \mathcal{K}).$$

So, its suffices to prove that

$$\max_i \text{osc}(V_k \circ \gamma_k^{-1}(t) - V^i \circ \gamma^{-1}(t)) \to 0, k \to \infty$$

and

$$\max_i \text{osc}(\Delta^0(K_k, \gamma_k^{-1}) - \Delta^0(K, \gamma^{-1})) \to 0, k \to \infty.$$
3.3. Part of the proof of Theorem 3.7

The main theorem of this paper is a consequence of the following lemma.

**Lemma 3.8 (Main Lemma).** Let \( \phi_{(H,\lambda)} \) be a sequence of symplectic isotopies such that \( \phi_{(H,\lambda)} \longrightarrow \text{Id} \). If \( (H_1,\lambda_1) \xrightarrow{L^\infty} (H,\lambda) \), then \( (H,\lambda) = (0,0) \).

Using the main lemma we can finish the proof of Theorem 3.7 as follows. Consider two sequences \( \phi_{(H,\lambda)} \) and \( \phi_{(K,\alpha)} \) of symplectic isotopies, and \( \Phi = (\phi_t) \in \mathcal{P}(\text{Homeo}(M), \text{id}) \) such that

\[
\phi_{(H,\lambda)} \xrightarrow{C^0} \Phi \xleftarrow{C^0} \phi_{(K,\alpha)},
\]

\[
(H_1,\lambda_1) \xrightarrow{L^\infty} (H,\lambda),
\]

and

\[
(K_1,\alpha_1) \xrightarrow{L^\infty} (K,\alpha),
\]

as indicated in Theorem 3.7.

By assumption, both sequences \( (\phi_{(H,\lambda)}), (H,\lambda) \) and \( (\phi_{(K,\alpha)}), (K,\alpha) \) converge in the topological group \( \text{GSSympeo}(M,\omega,g) \). This implies that the product \( (\phi_{(H,\lambda)}), (H,\lambda)) \ast (\phi_{(K,\alpha)}), (K,\alpha))^{-1} \) converges in \( \text{GSSympeo}(M,\omega,g) \). More precisely, the product \( (\phi_{(H,\lambda)}), (H,\lambda)) \ast (\phi_{(K,\alpha)}), (K,\alpha))^{-1} \) converges to \( (\Phi, (H,\lambda)) \ast (\Phi, (K,\alpha))^{-1} \) in \( \text{GSSympeo}(M,\omega,g) \), and by definition of the product and the inversion in \( \text{GSSympeo}(M,\omega,g) \), we have

\[
(\Phi, (H,\lambda)) \ast (\Phi, (K,\alpha))^{-1} = (\Phi, (H,\lambda)) \ast (\Phi^{-1}, (-K \circ \Phi - \Delta^0(\alpha,\Phi), -\alpha)) = (\text{Id}, (H - K - \Delta^0(\alpha,\Phi) \circ \Phi^{-1} - \Delta^0(\alpha,\Phi^{-1}), \lambda - \alpha)).
\]

Applying the main lemma (Lemma 3.8), we will deduce that:

\[
(H - K - \Delta^0(\alpha,\Phi) \circ \Phi^{-1} - \Delta^0(\alpha,\Phi^{-1}), \lambda - \alpha) = (0,0),
\]

i.e.

\[
H^t - K^t - \Delta^0(\alpha,\Phi) \circ \Phi^{-1} - \Delta^0(\alpha,\Phi^{-1}) = 0,
\]

and

\[
\lambda_t = \alpha_t,
\]

for each \( t \). Therefore, to have a complete proof of Theorem 3.7, we will prove in addition that \( H^t - K^t = 0 \), for each \( t \). This will follow as a consequence of Lemma 3.1 found in [6].

The proof of the main lemma is quite technical. Thus, we will process step by step to check it.

4. The null symplectic set

Following Buhovsky-Seyfaddini [7] we define the null symplectic set as follows.

**Definition 4.1.** We define the null symplectic set as the set of all the pairs \( (U, \mathcal{H}) \in \mathcal{I}(M,\omega,g)_0 \) such that \( (\text{Id}, (U, \mathcal{H})) \in \text{GSSympeo}(M,\omega,g) \).

In the following, we denote by \( \mathcal{G}^\infty \) the null symplectic set, and by \( \mathcal{G}^0 \) the set of time-independent elements of \( \mathcal{G}^\infty \).

**Lemma 4.2.** The spaces \( \mathcal{G}^\infty \) and \( \mathcal{G}^0 \) have the following properties:

1. \( \mathcal{G}^\infty \) is closed under the sum operator and minus operator.
2. \( \mathcal{G}^\infty \) is closed in \( L^\infty \) topology and \( \mathcal{G}^0 \) is closed in \( C^0 \) topology.
3. Let \( (U, \mathcal{H}) \in \mathcal{G}^\infty \), and \( \gamma : [0,1] \to [0,1] \) be a smooth increasing function. If \( U^\gamma \) is the continuous function \( (t,x) \mapsto \gamma(t) U_{\gamma(t)}(x) \), and \( \mathcal{H}^\gamma \) is the continuous family of harmonic 1-forms \( t \mapsto \gamma(t) \mathcal{H}_{\gamma(t)} \), then the element \( (U^\gamma, \mathcal{H}^\gamma) \) belongs to \( \mathcal{G}^\infty \).

Imhotep J.
4. \(G^0\) is a vector space over \(\mathbb{R}\).
5. If \((U, H) \in G^0\), then for any \(\rho \in \text{Symp}_0(M, \omega)\) we have
\[
\rho^*((U, H)) := (U \circ \rho + \square_t(H, \Phi = (\phi^t)), H) \in G^0,
\]
where \(\square_t(H, \Phi = (\phi^t))\) stands for the function \(x \mapsto \int_0^1 H_t(\phi^s) \circ \phi^t ds(x)\) normalized, and \(\Phi\) is any symplectic isotopy from the identity to \(\rho\).

**Proof.** For (1), let \((U, H), (V, K) \in G^\infty\). By definition, there exists a \(D^2\)-Cauchy sequence \((U^i, H^i) \in \mathcal{L}(M, \omega, g)\) such that the sequence of symplectic isotopies \(\phi(U^i, H^i)\) converges to the identity and \((U^i, H^i) \xrightarrow{L^\infty} (U, H)\). Compute,
\[
(U, H) = (-U - \Delta^0(H, Id), -H),
\]
where
\[
\Delta^0(H, Id) = \lim_{L^\infty}(\Delta(H, \phi(U^i, H^i))),
\]
vanishes identically since according to the continuity lemma from [6] (Lemma 3.1, [6]) the quantity \(\max_i, \text{osc}(\Delta(H, \phi(U^i, H^i)))\) tends to zero when the sequence of isotopies \(\phi(U^i, H^i)\) converges uniformly to the identity. That is,
\[
(U, H) = (-U, -H),
\]
and then
\[
(Id, -(U, H)) = (Id, (-U, -H)) = (Id, (U, H)) \in G\text{Sympeo}(M, \omega, g).
\]

Analogously, one deduces from the above arguments that
\[
(Id, (U + V, H + K)) = (Id, (U, H)) \ast (Id, (V, K)) \in G\text{Sympeo}(M, \omega, g),
\]
i.e,
\[
(U, H) + (V, K) = (U + V, H + K) \in G^\infty.
\]

For (5), let \((U, H) \in G^0\), and let \(\rho \in \text{Symp}_0(M, \omega)\). We have to prove that
\[
\rho^*((U, H)) := (U \circ \rho + \square_t(H, \Phi = (\phi^t)), H) \in G^0.
\]

On can derive from Lee-Cartan’s formulas that the element \(\rho^*\) \((U^i, H^i)\) does not depend on the choice of any isotopy from the identity to \(\rho\) (see [10]). Since \((U, H) \in G^0\), we derive that \(\square_t(H, \Phi = (\phi^t))\) is time-independent. So, the element \(\rho^*\) \((U, H)\) does not depend on \(t\). It remains to prove that \(\rho^*\) \((U, H)\) lies in \(G^\infty\). By assumption, there exists a \(D^2\)-Cauchy sequence \((U^i, H^i)\) such that \((U^i, H^i) \xrightarrow{L^\infty} (U, H)\), and the sequence of symplectic isotopies \(\phi(U^i, H^i)\) converges in \(d\) to the constant path identity. It is not too hard to see that the element \(\rho^*\) \((U^i, H^i)\) is the generator of symplectic isotopies \(\rho^{-1} \circ \phi(U^i, H^i) \circ \rho\), and the latter sequence converges in \(d\) to the constant path identity since \(d\) is bi-invariant. To conclude, we may prove that \(\rho^*\) \((U^i, H^i)\) \(\xrightarrow{L^\infty} \rho^*\) \((U, H)\) in \(\mathcal{L}(M, \omega, g)\). To that end, let’s compute
\[
\rho^*\) \((U^i, H^i)\) = (U^i \circ \rho + \square_t(H^i, \Phi = (\phi^t)), H^i),
\]
and
\[
\rho^*\) \((U, H)\) = (U \circ \rho + \square_t(H, \Phi = (\phi^t)), H).
\]

It is easy to see that
\[
\max_i |H^i_t - H_t| \to 0, i \to \infty,
\]
\[
\max_i (\text{osc}(U^i_t \circ \rho - U_t \circ \rho)) \to 0, i \to \infty,
\]
and
\[
\max_i (\text{osc}(\square_t(H^i - H, \Phi = (\phi^t)))) \leq 2diam(M) \left(1 + \sup_{t,s} |D\phi^t(\gamma_{st}(s))|\right) \max_{t \in [0,1]} |H_t - H^i_t|.
\]

Imhotep J.
where $D\phi^t$ stands for the tangent map of $\phi^t$, $y_0 \in M$ and $\gamma_{y_0}$ is a minimizing geodesic such that $\gamma_{y_0}(1) = y_0$ (see [6, Lemma 3.1]).

For (2), consider the continuous map

$$P_{r1} : GSSympeo(M, \omega, g) \to \mathcal{P}(Homeo(M), id),$$

$$(\xi, (F, \lambda)) \mapsto \xi.$$

The space $P_{r1}^{-1}(\{Id\})$ is a closed subset of $GSSympeo(M, \omega, g)$ since the map is continuous with respect of the symplectic topology on the space $GSSympeo(M, \omega, g)$ and the $C^0$–topology on the space $\mathcal{P}(Homeo(M), id)$. Of course, the restriction of symplectic topology to the subset $P_{r1}^{-1}(\{Id\})$ reduces to the $L^\infty$–topology. Hence, $G^\infty$ is closed with respect to the $L^\infty$–topology. The same arguments provide that $G^0$ is closed with respect to the $C^0$–topology. The item (4) follows from the definition of $G^0$.

For (3), let $(U, H) \in G^\infty$. By definition, there exists a $D^2$–Cauchy sequence $(U^i, H^i) \subset \mathfrak{T}(M, \omega, g)$ such that the sequence of symplectic isotopies $\phi_{(U^i, H^i)}$ converges to the identity and

$$(U^i, H^i) \overset{L^\infty}{\to} (U, H).$$

Consider a smooth increasing function $\gamma : [0, 1] \to [0, 1]$. If necessary, assume that the first derivatives of $\gamma$ are bounded. Then, the reparameterized sequence of symplectic paths $\phi_{(U^i, H^i)}^{\gamma(t)} : t \mapsto \phi_{(U^i, H^i)}^{\gamma(t)}$ converges in $\tilde{d}$ to the constant path identity, and the latter reparameterized sequence is generated by the sequence of generators $(U^i, H^i)$. Then, $(U^i, H^i)$ and $H^i(t) = \gamma(t)H^i_{\gamma(t)}$ for each $i$ and all $t$. But,

$$\max_i osc(U^i(t) - U^i_0) + \max_i |H^i(t) - H^i_0| \leq \max_i osc(U^i_0 - U^i) \sup_i |\gamma(t)|$$

$$+ \max_i |H^i - H^i_0| \sup_i |\gamma(t)|,$$

where each term in the right-hand side tends to zero when $i$ goes to infinity since $\sup_i |\gamma(t)| < \infty$.

This completes the proof.

Lemma 4.3. Let $(F, \lambda) \in G^\infty$. If $(F, \lambda) \in \mathfrak{T}(M, \omega, g)$, then $(F, \lambda) = (0, 0)$.

**Proof.** Since $(F, \lambda) \in G^\infty$ and $(F, \lambda) \in \mathfrak{T}(M, \omega, g)$, we derive that there exists a sequence $(U^i, H^i) \subset \mathfrak{T}(M, \omega, g)$ such that $\phi_{(U^i, H^i)} \overset{C^0}{\to} Id$, and $D_0((U^i, H^i), (F, \lambda)) \to 0, i \to \infty$. But, the latter convergences imply that $\overset{L^\infty}{\to} (\phi_{(F, \lambda)}^{-1} \circ \phi_{(U^i, H^i)}) \to 0, i \to \infty$, (see [13, Lemma 3.4]). It follows from the above arguments that we have simultaneously the following convergences

$$\phi_{(U^i, H^i)} \overset{C^0}{\to} Id,$$

$$\overset{L^\infty}{\to} (\phi_{(F, \lambda)}^{-1} \circ \phi_{(U^i, H^i)}) \to 0, i \to \infty.$$  

(4.1)

(4.2)

According to Corollary 3.8 in [13] or Theorem 13 in [5], equations (4.1) and (4.1) simultaneously impose that the symplectic isotopy $\phi_{(F, \lambda)}$ is the constant path identity, i.e. $\phi_{(F, \lambda)} = Id$. This in turn implies that $(F, \lambda) = (0, 0)$. This completes the proof.

Lemma 4.4. Let $(U, H) \in G^\infty$. Then for almost each $t \in [0, 1]$, the time-independent element $(V, K) := (U_t, H_t)$ lies in $G^0$.
Lemma 4.5. If $(U, \mathcal{H}) \in \mathcal{G}^0$, then $\mathcal{H} = 0$.

**Proof.** Let $(U, \mathcal{H}) \in \mathcal{G}^0$. We will process by contradiction. Assume that $\mathcal{H} \neq 0$.

- **Step (1).** By definition of $\mathcal{G}^0$, we can find a $D^2$-Cauchy sequence $(U^i, \mathcal{H}^i)$ in $\mathfrak{T}(M, \omega, g)$ such that

$$\phi(U^i, \mathcal{H}^i) \xrightarrow{\mathcal{d}} \text{Id},$$

and

$$(U^i, \mathcal{H}^i) \xrightarrow{L^\infty} (U, \mathcal{H}).$$

Hodge’s decomposition theorem of symplectic isotopies found in [4] implies

$$\phi(U^i, \mathcal{H}^i) = \rho_i \circ \Psi_i,$$

where $\rho_i$ is the harmonic isotopy generated by $(0, \mathcal{H}^i)$, and $\Psi_i$ is a Hamiltonian isotopy. From the definition of the elements of the set $\mathcal{G}^\infty$, we derive that $\mathcal{H} : t \mapsto \mathcal{H}_t$ is a continuous path in the space of harmonic 1-forms, and since $\mathcal{H}$ is time-independent, we derive that $\mathcal{H}$ is a smooth harmonic 1-form. Hence, it follows from the standard continuity theorem of ODE’s for Lipschitz vector fields that the sequence of symplectic isotopies generated by the sequence of harmonic 1-forms $(\mathcal{H}^i)$ converges in $\mathcal{d}$ to the symplectic isotopy generated by the harmonic 1-form $\mathcal{H}$ (see Lemma 3.4 found in [13], or [14] for a continuous version of the latter fact). That is, $(\rho_i)$ converges in $\mathcal{d}$ to the smooth harmonic flow $\phi((0, \mathcal{H})$. Knowing that $\phi(U^i, \mathcal{H}^i) \xrightarrow{\mathcal{d}} \text{Id}$, $\rho_i \xrightarrow{\mathcal{d}} \phi((0, \mathcal{H})$, and

$$\Psi_i := \rho_i^{-1} \circ \phi(U^i, \mathcal{H}^i),$$

for each $i$, we derive that the sequence $\Psi_i$ converges in $\mathcal{d}$ to a continuous isotopies $\Psi = (\psi_i)$. Moreover, we derive from the bi-invariance of the metric $\mathcal{d}$ that,

$$d(\Psi_i, \phi((0, \mathcal{H})) \leq d(\Psi_i, \phi(U^i, \mathcal{H}^i)) + d(\rho_i^{-1} \circ \phi(U^i, \mathcal{H}^i), \rho_i^{-1}) + d(\rho_i^{-1}, \phi((0, \mathcal{H})$$

for each $i$. The right-hand side tends to zero when $i$ goes to infinity, i.e. $\Psi = \phi((0, \mathcal{H})^{-1}$.

Imhotep J.
• Step (2). Since $\widetilde{Cal}(\phi_{(0,H)}) = [H]$ is non trivial, we derive from the arguments provided in Section 2.4 of the present paper that the flux of $\phi_{(0,H)}$ viewed as a volume preserving isotopy is non trivial, i.e. $\widetilde{\Phi}(\phi_{(0,H)}) \neq 0$. Therefore, it follows from Fathi’s duality result (see Proposition 2.1 of the present paper) that the Fathi mass flow of the path $\phi_{(0,H)}$ is non trivial, i.e. $\widetilde{\Phi}(\phi_{(0,H)}) \neq 0$. On the other hand, since by construction the path $\Psi$ is a continuous Hamiltonian flow in the sense of Oh-Müller [12], one derives from the continuity of the mass flow with respect to the $C^0$-topology that $\widetilde{\Phi}(\Psi) = 0$. But, it follows from step (1) that $\Psi = \phi^{-1}_{(0,H)}$, i.e.

\[ 0 = \widetilde{\Phi}(\Psi) = \widetilde{\Phi}(\phi^{-1}_{(0,H)}) = -\widetilde{\Phi}(\phi_{(0,H)}) \neq 0, \]

Equation (4.3) contradicts itself. This completes the proof.

Lemma 4.6. If $(F,K) \in G^0$, then $(F,K) = (0,0)$.

Proof. The proof follows closely that of Buhovsky-Seyfaddini for the similar statement in Hamiltonian case [7]. Assume that $(F,K) \in G^0$, and derive from Lemma 4.5 that $K$ is identically zero. It remains to prove that $F$ is identically zero. This is a verbatim repetition of arguments of Lemma 10 in [7].

Theorem 4.7. If $(Id,(F,\lambda)) \in GSSympeo(M,\omega,g)$, then $(F,\lambda) = (0,0)$.

Proof. Assume that $(Id,(F,\lambda)) \in GSSympeo(M,\omega,g)$. Then, $(F,\lambda) \in G^\infty$. From Lemma 4.4 we derive that for almost any $t \in [0,1]$, the time-independent element $(V,K) = (F,\lambda)$ lies inside $G^0$. Therefore, by Lemma 4.6 for such value of $t$, the element $(F,\lambda)$ vanishes, i.e.

\[ G^\infty = \{(0,0)\}. \]

This completes the proof.

Proof. The end of the proof of Theorem 3.7

Assume that both sequences $(\phi_{(H_{1},\lambda_{1})},(H_{1},\lambda_{1}))$ and $(\phi_{(K_{1},\alpha_{1})},(K_{1},\alpha_{1}))$ converge in the topological group $GSSympeo(M,\omega,g)$ to $(\Phi,(H,\lambda))$ and $(\Phi,(K,\alpha))$ respectively. Thus, the product $(\phi_{(H_{1},\alpha_{1})}),(H_{1},\alpha_{1})) \ast (\phi_{(K_{1},\alpha_{1})},(K_{1},\alpha_{1}))$ converges to $(\Phi,(H,\lambda)) \ast (\Phi^{-1},(K,\alpha) - \Delta^0(\alpha,\Phi),-(\alpha))$ in $GSSympeo(M,\omega,g)$. By definition of product in $GSSympeo(M,\omega,g)$, we have

\[ (\Phi,(H,\lambda)) \ast (\Phi^{-1},(K,\alpha) - \Delta^0(\alpha,\Phi),-(\alpha)) \]

\[ = (Id,(H - K - \Delta^0(\alpha,\Phi) \circ \Phi^{-1} - \Delta^0(\alpha,\Phi^{-1}),\lambda - \alpha)). \]

Applying theorem 4.7 we deduce that we may have

\[ (H - K - \Delta^0(\alpha,\Phi) \circ \Phi^{-1} - \Delta^0(\alpha,\Phi^{-1}),\lambda - \alpha) = (0,0), \]

which in turn yields $H^t - K^t - \Delta^0(\alpha,\Phi) \circ \Phi^{-1} - \Delta^0(\alpha,\Phi^{-1}) = 0$ and $\lambda = \alpha_t$ for each $t$. Using the fact that

\[ \Delta^0(\alpha,\Phi) \circ \Phi^{-1} = \lim_{L \to \infty} \{\tilde{\Delta}(\alpha,\phi_{(K,\alpha)}) \circ \phi^{-1}_{(H,\alpha)}\}, \]

\[ \Delta^0(\alpha,\Phi) = \lim_{L \to \infty} \{\tilde{\Delta}(\alpha,\phi_{(H,\alpha)})\}, \]

one checks that :

\[ \Delta^0(\alpha,\Phi) \circ \Phi^{-1} = \lim_{L \to \infty} \{\tilde{\Delta}(\alpha,\phi_{(K,\alpha)}) \circ \phi^{-1}_{(H,\alpha)} + \tilde{\Delta}(\alpha,\phi^{-1}_{(H,\alpha)})\} = \lim_{L \to \infty} \{\tilde{\Delta}(\alpha,\sigma_i)\} \]
since we always have
\[\tilde{\Delta}(\alpha, \sigma_i) = \tilde{\Delta}(\alpha, \phi^1_{(K_i, \lambda_i)}) \circ \phi^{-1}_{(H_i, \lambda_i)} + \tilde{\Delta}(\alpha, \phi^{-1}_{(H_i, \lambda_i)})\]
for each \(i\) where \(\sigma^t_i = \phi^t_{(K_i, \lambda_i)} \circ \phi^{-1}_{(H_i, \lambda_i)}\). Applying the continuity lemma from [6] (Lemma 3.1,[6]), we derive that
\[\lim_{L^\infty} \tilde{\Delta}(\alpha, \sigma_i) = 0,\]
since the sequence of isotopies \(\sigma_i\) converges in \(d\) to constant path identity. This yields
\[H^t = K^t,\]
for all \(t\). This completes the proof.

References


Augustin Banyaga

e-mail: banyaga@math.psu.edu

Imhotep J.